

# Enhanced symmetries of gauge theory and resolving the spectrum of local operators

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## Abstract

Enhanced global non-abelian symmetries at zero coupling in Yang Mills theory play an important role in diagonalising the two-point functions of multi-matrix operators. Generalised Casimirs constructed from the iterated commutator action of these enhanced symmetries resolve all the multiplicity labels of the bases of matrix operators which diagonalise the two-point function. For the case of  $U(N)$  gauge theory with a single complex matrix in the adjoint of the gauge group we have a  $U(N)^{\times 4}$  global symmetry of the scaling operator at zero coupling. Different choices of commuting sets of Casimirs, for the case of a complex matrix, lead to the restricted Schur basis previously studied in connection with string excitations of giant gravitons and the Brauer basis studied in connection with brane-anti-brane systems. More generally these remarks can be extended to the diagonalisation for any global symmetry group  $G$ . Schur-Weyl duality plays a central role in connecting the enhanced symmetries and the diagonal bases.

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# 1 Introduction

Local gauge invariant operators in a conformal field theory correspond to physical states. Understanding the spectrum of states from the point of view of the CFT yields information about spacetime physics via the AdS/CFT duality [1, 2, 3]. The two-point functions of the CFT give an inner product on the states. The diagonalisation of this inner product is a valuable tool in the detailed understanding of spacetime physics. States created by holomorphic multi-traces of one complex matrix are half-BPS. The diagonalisation in this sector has interesting connections with free fermions [4, 5], giant gravitons [6], LLM geometries [7] and black hole physics [8] in the context of  $U(N)$  gauge theory with  $\mathcal{N} = 4$  supersymmetry and its string dual in  $AdS_5 \times S^5$ .

Recent progress on the diagonalisation of the two-point functions of gauge invariant multi-matrix operators [9, 10, 11, 12] has generalised earlier work on the holomorphic sector of a single complex matrix [4, 13]. In this paper, we will explain how the labels appearing in the diagonal bases are related to Casimirs constructed from Noetherian symmetries in the zero coupling limit. The construction of these diagonal bases heavily uses symmetric group data, or related finite algebras such as Brauer algebras. The Noetherian symmetries are unitary Lie group symmetries, e.g products of  $U(N)$  or  $U(N^2)$ . Schur-Weyl duality explains the relation between these unitary symmetries and the symmetric group construction of the diagonal bases.

The first case we consider is the sector of one complex matrix. In this case we can use two diagonalisation methods : the Brauer Basis [9] and the restricted Schur basis [11]. The second case we will consider is the case of the holomorphic sector of  $M$  complex matrices. We also sketch the application of enhanced symmetries in the case of the diagonalisation with general global symmetry  $G$  given in [12].

## 1.1 Schur-Weyl duality and enhanced symmetries

The diagonal basis for the holomorphic sector of one complex matrix  $X$ , which is relevant for the half-BPS representations of  $\mathcal{N} = 4$  SYM, is labelled by Young diagrams  $R$  of  $U(N)$ , i.e those with first column no longer than  $N$ . We have  $\mathcal{O}_R = \chi_R(X)$ . For operators made from  $n$  fields  $X$ , the Young diagrams have  $n$  boxes. This means that  $R$  is a partition of  $n$  which we write as  $R \vdash n$  and the constraint on the columns is expressed as  $c_1(R) \leq N$ . These Young diagrams are also associated with representations of the symmetric group  $S_n$  of all permutations of  $n$  objects. To understand the role of  $U(N)$  and  $S_n$  it is useful to view the  $X$  as an operator acting on an  $N$ -dimensional vector space  $V$ . It can be extended to  $\mathbf{X} = X \otimes X \cdots \otimes X$  acting on  $V^{\otimes n}$ . The basic reason for the appearance of  $S_n$  is that elements  $\sigma$  of the symmetric group organise the space of multi-traces. The multi-trace operators can be written as  $tr_n(\sigma \mathbf{X})$  for  $\sigma \in S_n$  with  $tr_n$  denoting a trace in  $V^{\otimes n}$ . The role of  $R \vdash n$  with  $c_1(R) \leq N$  can be understood by thinking about the decomposition of  $V^{\otimes n}$  in terms of  $U(N) \times S_n$

$$V^{\otimes n} = \bigoplus_R V_R^{U(N)} \otimes V_R^{S_n} \quad (1)$$

This equation is called Schur-Weyl duality and follows from the fact that the algebra of operators commuting with  $U(N)$  in  $V^{\otimes n}$  is  $\mathbb{C}(S_n)$ , the group algebra of  $S_n$ . Conversely the algebra commuting with  $S_n$  is the universal enveloping algebra of  $U(N)$ . For a review of results in two and four dimensional Yang Mills which rely on (1) and its generalisations, see [14].

A critical physicist might argue that the above account raises a small mystery. The  $U(N)$  gauge symmetry of Yang Mills is not a dynamical symmetry acting on the physical spectrum. On the other hand a symmetry that organises operators is a dynamical symmetry. Indeed operators in conformal field theory correspond to states and an algebra that organises the operators organises the states. The resolution is that, at zero coupling the global part of the  $U(N)$  gauge symmetry is part of a bigger symmetry which includes  $U(N) \times U(N)$  acting separately on the lower and upper indices of  $X_j^i$ . This is explained in more detail in section 2.1. This enhanced symmetry commutes with the classical dilatation operator and hence leaves unchanged the space of operators made from a fixed number of  $X$ . It turns out that the enhanced symmetries can be used to construct Casimirs which act on  $\mathcal{O}_R$  to give eigenvalues  $C_2(R), C_3(R) \cdots$ . This remark was essentially already contained in [4] where it was discussed in the context of a reduced matrix model on  $S^3 \times R$  where the complex field has a mass term. Here we will not do the dimensional reduction and will discuss global symmetries of the four dimensional theory. The Casimirs constructed in the context of the holomorphic  $X$  sector have been viewed as charges in spacetime [15]. Viewing these charges as physical observables in the dual spacetime can be used to justify interest in the basis  $\mathcal{O}_R$ . From a technical point of view, it is worth observing that states which have distinct eigenvalues of a complete set of Casimirs form an orthogonal basis by a standard quantum mechanics argument. We may view the fact these diagonal bases form eigenstates

of the Casimirs as evidence that they are useful for physics. In the case of the more general diagonalisations involving multiple matrices and global symmetries, the diagonal bases found recently use, along with Young diagram labels, some more subtle group theoretic labels from the world of symmetric groups and Brauer algebras. We find that Schur Weyl duality is a valuable guide which helps anticipate the type of Casimirs, to be constructed from enhanced symmetries of the Yang Mills Lagrangian at  $g_{YM}^2 = 0$ , relevant in these more general cases.

One immediate consequence of Schur-Weyl (SW) duality (1) is that Casimirs constructed from  $U(N)$  generators can be expressed in terms of elements of  $\mathbb{C}(S_n)$  (see equation (31)). This intimate connection between Casimirs and SW duality can be generalised in several directions. The classical Schur-Weyl duality above is a special case of a more general theorem called the double commutant theorem, which gives a similar decomposition for any space  $\mathcal{W}$  under the action of an algebra  $A$

$$\mathcal{W} = \bigoplus_{\Lambda} V_{\Lambda}^A \otimes V_{\Lambda}^{Com(A)} \quad (2)$$

In the above  $Com(A)$  is the commutant of  $A$ ,  $V_{\Lambda}^A$  is an irreducible representation (irrep) of  $A$ ,  $V_{\Lambda}^{Com(A)}$  is an irrep of the  $Com(A)$ . This implies that the multiplicity of each irrep of  $A$  is given by the dimension of a corresponding irrep of  $Com(A)$ . For details on the double commutant theorem (also called the double centraliser theorem), see section 1 of [16] or textbooks such as [17]. The following fact from double commutant theory will be useful. Suppose  $A$  acts on  $\mathcal{W}$  and has commutant  $Com(A)$ . Suppose further that  $A$  has a subalgebra  $B$ . Clearly  $Com(A)$  is a subalgebra of  $Com(B)$ . In this case suppose  $V_{\mu}^B$  appears in the sub-algebra decomposition of  $V_{\lambda}^A$  with multiplicity  $g_{\lambda\mu}$ . And further suppose that the decomposition of  $V_{\mu}^{Com(B)}$  in terms of the sub-algebra  $Com(A)$  contains  $V_{\lambda}^{Com(A)}$  with multiplicity  $g'_{\mu\lambda}$ . Then the useful result is  $g_{\lambda\mu} = g'_{\mu\lambda}$ . This implies that various group theoretic multiplicities defined in the world of symmetric groups have a dual meaning in the world of unitary groups and vice versa.

We will make extensive use of the above concept. In sections 3.2 and 4.1 we will be considering a diagonal basis of operators [9] in the sector of  $X, X^*$  given in terms of the Brauer algebras  $B_N(m, n)$  and its reduction to the subalgebra  $\mathbb{C}(S_m) \times \mathbb{C}(S_n)$ . The SW dual of  $B_N(m, n)$  is  $U(N)$  acting on  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ . So we expect the Casimirs to involve multiple copies of  $U(N)$ . All the relevant  $U(N)$  symmetries of the classical scaling operator are found in section 2.1. A different combination of the same  $U(N)$  symmetries is used to construct Casimirs which resolve the labels on the restricted Schur basis of operators given in [11] which involve the reduction of  $\mathbb{C}(S_{m+n})$  to  $\mathbb{C}(S_m) \times \mathbb{C}(S_n)$ . This is done in sections 3.3 and 4.2. Another case considered is the sector of  $M$  complex matrices where a diagonal basis of holomorphic operators was given in [10] which is  $U(M)$  covariant. The labels on the basis include a Clebsch multiplicity  $\tau$  which runs over the number of times the irrep  $\Lambda$  of  $S_n$  appears in the tensor product (sometimes called inner tensor product)  $R \otimes R$  of the irrep  $R$  of  $S_n$ . Now the tensor product  $R \otimes R$  is a representation of  $S_n \times S_n$  and the Clebsch reduction problem is equivalently the problem of decomposing the representations

of the product group in terms of the diagonal  $S_n$  subgroup. This inner tensor product problem of symmetric groups is known ( see e.g [18]) to be related by Schur Weyl duality to the embedding of  $U(N) \times U(N)$  inside  $U(N^2)$  (in fact more generally to the embedding of  $U(M) \times U(N)$  to  $U(MN)$ ). We will describe the relevant  $U(N^2)$  as symmetries constructed from the free field lagrangian in section 2. In sections 3.4 and 4.3 we will show how the generators of  $U(N^2)$  along with  $U(N)$  can be used to construct the invariant generalised Casimirs which distinguish the operators with different values of  $\tau$ .

In section 5 we give computations of some of the eigenvalues of these generalised Casimirs.

## 2 Enhanced Symmetries in zero coupling gauge theories

Zero coupling  $U(N)$  gauge theories have a large global symmetry group. The key features are best described in the simple example of a complex scalar transforming in the adjoint of the gauge group. We first describe a  $U(N)^{\times 4}$  symmetry whose diagonal subgroup acts the gauge transformation. The diagonal of course leaves all the gauge invariant operators unchanged. But the full group acts non-trivially on the spectrum of gauge invariant states. Then we describe a  $U(N^2) \times U(N^2)$  symmetry which contains the  $U(N)^{\times 4}$ . This symmetry essentially follows from the fact that we have  $N^2$  free fields.

### 2.1 One complex Matrix : $U(N)^{\times 4}$

In  $\mathcal{N} = 4$  SYM, the highest weight states are in 1-1 correspondence with local operators constructed from holomorphic multi-trace combinations of a single complex matrix  $X = \phi_1 + i\phi_2$ , where  $\phi_1, \phi_2$  are two of the six hermitian matrices in  $\mathcal{N} = 4$  SYM.

The action for the complex scalar coupled to Yang Mills is

$$\frac{1}{2} \int tr F_{\mu\nu} F^{\mu\nu} + tr D_\mu X D^\mu X^\dagger \quad (3)$$

We are interested in gauge invariant operators constructed from the scalar and their correlators at zero coupling. For these computations of correlators the relevant part of the action is

$$\int tr D_\mu X D^\mu X^\dagger \quad (4)$$

Working in the  $A_0 = 0$  gauge we can separate out the part of the action containing  $A_0$ , which acts as a Lagrange multiplier for the Gauss Law Constraint [19]. The matter coupling of  $A_0$  coming from  $D_0 X D_0 X^\dagger$  is

$$\int tr (A_0 X^\dagger \partial_0 X + A_0 X \partial_0 X^\dagger - A_0 \partial_0 X X^\dagger - A_0 \partial_0 X^\dagger X) \quad (5)$$

In canonical quantisation the Gauss Law matrix operator

$$\mathcal{G} = X^\dagger \partial_0 X + X \partial_0 X^\dagger - \partial_0 X X^\dagger - \partial_0 X^\dagger X \quad (6)$$

generates time independent gauge transformations. We will in fact be interested in the global transformations which are also independent of time, and are generated by

$$G_j^i = -i \int d^3x \mathcal{G}_j^i \quad (7)$$

The momenta conjugate to  $X_j^i$  and  $X_j^{\dagger i}$  are

$$\begin{aligned} \frac{\partial L}{\partial \partial_0 X_j^i} &= \partial_0 X^{\dagger j}_i = \Pi_{X_j^i} \\ \frac{\partial L}{\partial \partial_0 X_j^{\dagger i}} &= \partial_0 X^j_i = \Pi_{X^{\dagger j}_i} \end{aligned} \quad (8)$$

The canonical commutation relations are

$$\begin{aligned} [(\Pi_X)_i^j(x), X_q^p(0)] &= i \delta_i^p \delta_q^j \delta^3(x) \\ [(\Pi_{X^\dagger})_i^j(x), X_q^{\dagger p}(0)] &= i \delta_i^p \delta_q^j \delta^3(x) \end{aligned} \quad (9)$$

Corresponding to the four terms in (6) we will write the Gauss Law operator as

$$G = G_1 + G_2 + G_3 + G_4 \quad (10)$$

and they act as follows

$$\begin{aligned} [G_{1j}^i, X_q^{\dagger p}] &= [-i \int d^3x (X^\dagger \partial_0 X)_j^i(x), X_q^{\dagger p}] = \delta_j^p X_q^{\dagger i} \\ [G_{2j}^i, X_q^p] &= [-i \int d^3x (X \partial_0 X^\dagger)_j^i(x), X_q^p] = \delta_j^p X_q^i \\ [G_{3j}^i, X_q^{\dagger p}] &= [i \int d^3x (\partial_0 X X^\dagger)_j^i(x), X_q^{\dagger p}] = -\delta_q^i X_j^{\dagger p} \\ [G_{4j}^i, X_q^p] &= [i \int d^3x (\partial_0 X^\dagger X)_j^i(x), X_q^p] = -\delta_q^i X_j^p \end{aligned} \quad (11)$$

Equivalently we can express the commutation relations as follows

$$\begin{aligned} [tr(\epsilon_1 G_1), X_q^{\dagger p}] &= (\epsilon_1 X^\dagger)_q^p \\ [tr(\epsilon_2 G_2), X_q^p] &= (\epsilon_2 X)_q^p \\ [tr(\epsilon_3 G_3), X_q^{\dagger p}] &= -(X^\dagger \epsilon_3)_q^p \\ [tr(\epsilon_4 G_4), X_q^p] &= -(X \epsilon_4)_q^p \end{aligned} \quad (12)$$

The commutation relations of the  $G_a$  (  $a = 1 \cdots 4$  ) are those of the generators of  $U(N)^{\times 4}$

$$[(G_a)_j^i, (G_b)_l^k] = \delta_{ab}((G_a)_l^i \delta_j^k - (G_a)_j^k \delta_l^i) \quad (13)$$

Exponentiating these actions we get the action of four copies of  $U(N)$ , respectively

$$\begin{aligned} X^\dagger &\rightarrow U_1 X^\dagger \\ X &\rightarrow U_2 X \\ X^\dagger &\rightarrow X^\dagger U_3^\dagger \\ X &\rightarrow X U_4^\dagger \end{aligned} \tag{14}$$

Given these actions it is natural to rename

$$\begin{aligned} G_1 &= G(\mathcal{L}, X^\dagger) \\ G_2 &= G(\mathcal{L}, X) \\ G_3 &= G(\mathcal{R}, X^\dagger) \\ G_4 &= G(\mathcal{R}, X) \end{aligned} \tag{15}$$

where  $\mathcal{L}$  denotes left action and  $\mathcal{R}$  denotes right action. When we set  $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4$  we have the adjoint action on  $X, X^\dagger$  which is the global gauge symmetry. It constrains local operators to be constructed from traces. Note however that setting  $\epsilon_2 = \epsilon_3$  we have  $\epsilon_2(G_2 + G_3)$  which is a symmetry of the Lagrangian. Likewise setting  $\epsilon_1 = \epsilon_4$  we have  $\epsilon_1(G_1 + G_4)$  which is also a symmetry. In fact, for the comparison to the global time Hamiltonian of AdS, we are interested in the scaling operator

$$L_0 = \int d^3x \left( X_j^i (\Pi_X)_i^j + X_j^\dagger (\Pi_{X^\dagger})_i^j \right) = \int d^3x \operatorname{tr} (X \Pi_X + X^\dagger \Pi_{X^\dagger}) \tag{16}$$

By using (9) it is easy to check that

$$\begin{aligned} [G_a, \int d^3x \operatorname{tr} (X \Pi_X)] &= 0 \quad \text{for } a = 1 \cdots 4 \\ [G_a, \int d^3x \operatorname{tr} (X^\dagger \Pi_{X^\dagger})] &= 0 \quad \text{for } a = 1 \cdots 4 \end{aligned} \tag{17}$$

So we have, in the zero coupling limit, a symmetry of  $U(N) \times U(N) \times U(N) \times U(N)$  of the scaling operator. The diagonal of these actions is nothing but the gauge group action under which all local traced operators are going to be invariant. However we are also interested in the full  $U(N)^{\times 4}$  in organising the gauge invariant operators into a basis. As an aside note that if we take the parameters  $\epsilon_j^i$  to be general complex numbers we actually have  $GL(N, \mathbb{C})^4$ , whereas to get  $U(N)^{\times 4}$  we let  $\epsilon$  be hermitian. For our applications of symmetry groups organising the diagonal bases for two-point functions, it suffices to use unitary groups rather than the complex form, since the unitary groups have the same commutants as the general linear groups.

## 2.2 $U(N^2) \times U(N^2)$ symmetry

The free action for a complex matrix in fact has a  $U(N^2)$  symmetry. This is clear by writing

$$\int d^4x \operatorname{tr} (\partial_0 X \partial_0 X^\dagger) = \int d^4x \partial_0 X_j^i \partial_0 X_j^{i*} = \int d^4x \partial_0 X_A \partial_0 X_A^* \tag{18}$$



where  $A = (i, j)$  is a composite index, taking  $N^2$  values as  $i, j$  run over  $N$  values. In canonical quantisation we have generators

$$E_{lm}^{jk}(X) = -i \int d^3x X_l^k \Pi_{X_j^m} \quad (19)$$

They transform the  $X$  as

$$[E_{lm}^{jk}, X_q^p] = \delta_m^p \delta_q^j X_l^k \quad (20)$$

Likewise we have a  $U(N^2)$  symmetry for the  $X^\dagger$ .

$$E_{lm}^{jk}(X^\dagger) = -i \int d^3x X_l^{\dagger k} \Pi_{X_j^{\dagger m}} \quad (21)$$

One checks that these are generators of a  $U(N^2) \times U(N^2)$  algebra which leaves the classical scaling operator  $tr(X\Pi_X)$  invariant. Each  $U(N^2)$  has a  $U(N) \times U(N)$  symmetry, so the  $U(N)^{\times 4}$  is a subgroup of the  $U(N^2) \times U(N^2)$ .

For  $M$  complex matrices,  $X_1, X_2, \dots, X_M$  we have  $m$  copies  $(U(N^2) \times U(N^2))^{\times M}$  generated by  $(E_{lm}^{jk}(X_1), E_{lm}^{jk}(X_1^\dagger)), \dots, (E_{lm}^{jk}(X_M), E_{lm}^{jk}(X_M^\dagger))$ . Defining in this case

$$E_{lm}^{jk} = E_{lm}^{jk}(X_1) + \dots + E_{lm}^{jk}(X_M) \quad (22)$$

This will act as

$$[E_{lm}^{jk}, (X_a)_q^p] = \delta_m^p \delta_q^j (X_a)_l^k \quad (23)$$

for any  $a = 1 \dots M$ .

## 2.3 Enhanced symmetries of $\mathcal{N} = 4$ SYM

The full SYM has a product group symmetry  $U(N)^{\times 4} \times U(N)^{\times 4} \dots \times U(N)^{\times 4}$  consisting of 8 factors, of which 4 act on bosonic fields and 4 act on fermionic fields. There are 3 bosonic  $U(N)^{\times 4}$  coming from 3 complex scalars. There is 1 bosonic  $U(N)^{\times 4}$  for the gauge field, which is easiest to describe in the light-cone gauge where the kinetic term takes the form  $\int tr \partial_+ A_z \partial_- A_{\bar{z}}$  and a discussion very similar to the above treatment of a complex scalar field can be done. These four copies of  $U(N)^{\times 4}$  are subgroups of four copies of  $U(N^2) \times U(N^2)$ . In the fermionic sector, there are four two-component Weyl fermions with kinetic term  $i \int tr \partial_m \bar{\Psi}_a \bar{\sigma}^m \Psi_a$  with the flavour index  $a$  taking values from 1 to 4. For each fixed flavour index  $a$ , we have charges

$$\begin{aligned} (G_\alpha(\mathcal{L}, \Psi))_k^i &= (\Psi_\alpha)_j^i \Pi_{(\Psi_\alpha)_j^k} \\ (G_\alpha(\mathcal{R}, \Psi))_k^i &= \Pi_{(\Psi_\alpha)_i^j} (\Psi_\alpha)_k^j \end{aligned} \quad (24)$$

As  $\alpha$  takes two values, there are four commuting copies of  $U(N)$  for each of the four flavours. This gives four copies of  $U(N)^{\times 4}$  in the fermionic sector, which are subgroups of  $(U(N^2))^{\times 2}$

### 3 Casimirs of Enhanced symmetries and Diagonal bases

#### 3.1 Enhanced symmetry and Casimirs for Holomorphic sector of $X$

The operators  $\mathcal{O}_R = \chi_R(\mathbf{X})$  are eigenstates of the Casimirs constructed  $tr((Ad_{G_{\mathcal{L}}})^n)$  where  $G_{\mathcal{L}} = G(\mathcal{L}, X)$ . This remark was contained in [4] in the slightly different context of the reduction to quantum mechanics.

Recall some standard group theory of the generators  $E_j^i$  of  $U(N)$

$$[E_j^i, E_l^k] = \delta_j^k E_l^i - \delta_l^i E_j^k \quad (25)$$

Consider the vectors in the fundamental representation  $V$

$$(E)_j^i v^p = \delta_j^p v^i \quad (26)$$

On the tensor space  $V^{\otimes n}$  we have the action

$$\begin{aligned} & (E)_j^i (v^{p_1} \otimes v^{p_2} \cdots \otimes v^{p_n}) \\ &= E_j^i(v^{p_1}) \otimes v^{p_2} \otimes \cdots \otimes v^{p_n} + v^{p_1} \otimes E_j^i(v^{p_2}) \otimes \cdots \otimes v^{p_n} + \cdots + v^{p_1} \otimes v^{p_2} \otimes \cdots \otimes E_j^i(v^{p_n}) \end{aligned} \quad (27)$$

We know that the Casimir will be constant on the irreducible subspaces of  $V^{\otimes n}$ . The decomposition of  $V^{\otimes n}$  in terms of  $U(N) \times S_n$  is given by Schur-Weyl duality

$$V^{\otimes n} = \bigoplus_R V_R^{U(N)} \otimes V_R^{S_n} \quad (28)$$

The projector for  $R$  in terms of  $S_n$  group theory can be given as

$$p_R = \frac{d_R}{n!} \sum_{\sigma} \chi_R(\sigma) \sigma \quad (29)$$

Hence it follows that

$$\hat{C}_2 p_R(v^{p_1} \otimes v^{p_2} \cdots \otimes v^{p_n}) = C_2(R) p_R(v^{p_1} \otimes v^{p_2} \cdots \otimes v^{p_n}) \quad (30)$$

This can be seen to follow from (28) and the fact that  $p_R$  projects the direct sum to a fixed  $R$ , so the eigenvalue is the quadratic  $U(N)$  Casimir  $C_2(R)$ . Schur-Weyl duality also gives more information on  $C_2(R)$ .

It follows from Schur-Weyl duality (28) that the action of  $\hat{C}_2 = E_j^i E_i^j$  on  $V^{\otimes n}$ , which commutes with  $U(N)$ , must be expressible in terms of central elements of  $\mathbb{C}(S_n)$ . Since the Casimir commutes with  $U(N)$  and the commutant of  $U(N)$  in  $V^{\otimes n}$  is  $S_n$ , it follows that the Casimir is in  $\mathbb{C}(S_n)$ . Since it is constructed from generators of  $U(N)$  which commute with

$S_n$ , this means that it must be a central element, i.e something in  $\mathbb{C}(S_n)$  which commutes with all elements in  $\mathbb{C}(S_n)$ . Explicitly,

$$\begin{aligned}\hat{C}_2 &= Nn + \sum_{r \neq s} (rs) \\ &\equiv Nn + 2T_2\end{aligned}\tag{31}$$

We review the derivation in Appendix A. The consequence for eigenvalues is

$$\begin{aligned}C_2(R) &= Nn + 2 \frac{\chi_R(T_2)}{d_R} \\ &= Nn + \sum_i r_i(r_i - 2i + 1)\end{aligned}\tag{32}$$

Here  $d_R$  is the dimension of the symmetric group representation  $R$ , and the last line follows using a standard result on the characters of  $S_n$ . The above equation (32) plays an important role in the string theory of two dimensional Yang Mills [20]. To get the explicit formula (32) for eigenvalues  $C_2(R)$  it is useful to use (31).

The Schur Polynomial operators can be written as

$$\begin{aligned}\mathcal{O}_R \equiv \chi_R(\mathbf{X}) &= \frac{1}{n!} \sum_{\sigma} \chi_R(\sigma) \text{tr}_n(\sigma \mathbf{X}) \\ &= \frac{1}{d_R} \text{tr}_n(p_R \mathbf{X}) \\ &= \frac{1}{d_R} (p_R)_I^J (\mathbf{X})_J^I\end{aligned}\tag{33}$$

The last expression will be useful. Here  $I, J$  are multi-indices e.g  $I = (i_1, i_2, \dots, i_n)$ . The action of  $Ad_{(G_{\mathcal{L}})_j^i}$  on the upper indices of  $\mathbf{X}$  is the same as the action of  $E_j^i$  on  $V^{\otimes n}$ . Hence the iterated commutator action is

$$\begin{aligned}[(G_{\mathcal{L}})_j^i, [(G_{\mathcal{L}})_i^j, \mathcal{O}_R]] &= Ad_{(G_{\mathcal{L}})_j^i} Ad_{(G_{\mathcal{L}})_i^j} \mathcal{O}_R \\ &= \frac{1}{d_R} (p_R)_I^J (\hat{C}_2)_K^I (\mathbf{X})_J^K \\ &= \frac{1}{d_R} (\hat{C}_2 p_R)_K^J (\mathbf{X})_J^K \\ &= \frac{1}{d_R} C_2(R) (p_R)_K^J (\mathbf{X})_J^K \\ &= C_2(R) \mathcal{O}_R\end{aligned}\tag{34}$$

In the third line we used (30).

### 3.2 Enhanced symmetry and Casimirs : Brauer Basis for $X, X^\dagger$

Consider  $G_B \equiv G_2 + G_3$  which generates  $X \rightarrow UX, X^\dagger \rightarrow X^\dagger U^\dagger$  equivalently

$$X \rightarrow UX$$

$$X^* \rightarrow U^* X^* \quad (35)$$

On gauge invariant operators  $G_B = -G_1 - G_4$  because  $\sum_{a=1}^4 G_a$  generates the adjoint gauge transformation  $X \rightarrow UXU^\dagger$ ,  $X^\dagger \rightarrow UX^\dagger U^\dagger$ . Under the above action, the lower indices of  $X, X^*$  are inert while the upper indices transform as  $V \otimes \bar{V}$ , i.e the fundamental and its complex conjugate. Composite operators are built by considering  $m$  copies of  $X$  and  $n$  copies of  $X^*$ . It is useful to view these as operators of the form  $X \otimes X \cdots \otimes X \otimes X^* \otimes \cdots \otimes X^*$  acting on  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ . The Schur Weyl dual of the  $U(N)$  action on this space is the Brauer algebra  $B_N(m, n)$

The Brauer algebra can be used to construct a diagonal basis [9] of gauge invariant operators  $\mathcal{O}_{\alpha, \beta; i, j}^\gamma \equiv \text{tr}_{m, n}(Q_{\alpha, \beta; i, j}^\gamma(\mathbf{X} \otimes \mathbf{X}^*))$ . The label  $\gamma$  denotes an irrep of  $B_N(m, n)$ . The labels  $\alpha, \beta$  denote irreps. of  $S_m, S_n$  respectively. Equivalently  $(\alpha, \beta)$  is an irrep of the product group  $S_m \times S_n$ . We will often use the label  $A \equiv (\alpha, \beta)$  as an abbreviation and states will be denoted with  $m_A \equiv (m_\alpha, m_\beta)$ . The indices  $i, j$  here each run over the mutiplicity of  $A$  in the restriction of the irrep  $\gamma$  of  $B_N(m, n)$  to the irrep  $A$  of the subalgebra  $\mathbb{C}(S_m \times S_n)$ . Note we also often use  $i, j$  among  $U(N)$  indices : viewing the formulae in context should resolve any possible confusion. More on this basis of operators in the Appendix B.

We will show the following iterated commutator actions

$$\begin{aligned} \text{tr}(Ad_{G_B}^2) \mathcal{O}_{\alpha, \beta; i, j}^\gamma &= Ad_{(G_B)_j^i} Ad_{(G_B)_i^j} \mathcal{O}_{\alpha, \beta; i, j}^\gamma = C_2(\gamma) \mathcal{O}_{\alpha, \beta; i, j}^\gamma \\ \text{tr}(Ad_{G_2}^2) \mathcal{O}_{\alpha, \beta; i, j}^\gamma &= Ad_{(G_2)_j^i} Ad_{(G_2)_i^j} \mathcal{O}_{\alpha, \beta; i, j}^\gamma = C_2(\alpha) \mathcal{O}_{\alpha, \beta; i, j}^\gamma \\ \text{tr}(Ad_{G_3}^2) \mathcal{O}_{\alpha, \beta; i, j}^\gamma &= Ad_{(G_3)_j^i} Ad_{(G_3)_i^j} \mathcal{O}_{\alpha, \beta; i, j}^\gamma = C_2(\beta) \mathcal{O}_{\alpha, \beta; i, j}^\gamma \\ \text{tr}(Ad_{G_2}^2 Ad_{G_3}) \mathcal{O}_{\alpha, \beta; i, j}^\gamma &= Ad_{(G_2)_j^i} Ad_{(G_2)_k^j} Ad_{(G_3)_i^k} \mathcal{O}_{\alpha, \beta; i, j}^\gamma = C_{223}(\gamma; \alpha, \beta, i) \mathcal{O}_{\alpha, \beta; i, j}^\gamma \\ \text{tr}(Ad_{G_1}^2 Ad_{G_4}) \mathcal{O}_{\alpha, \beta; i, j}^\gamma &= Ad_{(G_1)_j^i} Ad_{(G_1)_k^j} Ad_{(G_4)_i^k} \mathcal{O}_{\alpha, \beta; i, j}^\gamma = C_{114}(\gamma; \alpha, \beta, j) \mathcal{O}_{\alpha, \beta; i, j}^\gamma \quad (36) \end{aligned}$$

The Casimirs  $C_2(\gamma)$  is a  $U(N)$  Casimir for the composite representation  $\gamma$ . The Casimirs  $C_2(\alpha)$  and  $C_2(\beta)$  are  $U(N)$  Casimirs for the representations  $\alpha$  with  $m$  boxes and  $\beta$  with  $n$  boxes. The Casimirs  $C_{114}(j), C_{223}(i)$  are less familiar. They can be related to elements of the Brauer algebra which are invariant under the  $\mathbb{C}(S_m \times S_n)$  subalgebra, and can be used to distinguish different copies of the same representation  $(\alpha, \beta)$  of  $\mathbb{C}(S_m \times S_n)$  appearing in the reduction of the representation  $\gamma$  of  $B_N(m, n)$ . Examples of these eigenvalues will be computed in section 5 .

Note that while  $Ad_{G_B}$  and the Casimir  $\text{tr}(Ad_{G_B} Ad_{G_B})$  leaves the action invariant, the action of  $Ad_{G_1}$  does not leave the action invariant. It is nevertheless useful in organising the operators at zero coupling, since it is a symmetry of the classical scaling operator. It is also worth noting that  $\text{tr} Ad_{G_1} Ad_{G_1}$  has the rather simple effect of scaling the action.

### 3.3 Enhanced symmetry and Casimirs : Restricted Schur Basis for $X, X^\dagger$

In [11], another complete set of gauge invariant operators constructed from  $X$  and  $X^\dagger$  was proposed, where the symmetric group was used instead of the Brauer algebra<sup>1</sup>. Here we have a basis  $\mathcal{O}_{R_1, R_2; i, j}^R \equiv \text{tr}_{m, n}(Q_{R_1, R_2; i, j}^R(\mathbf{X} \otimes \mathbf{X}^\dagger))$ .  $R$  is an irreducible representation of the symmetric group  $S_{m+n}$ , and  $R_1$  and  $R_2$  are irreducible representations of  $S_m$  and  $S_n$ . The indices  $i, j$  here each run over the multiplicity of the irrep  $R_1 \otimes R_2$  of  $\mathbb{C}(S_m \times S_n)$  in the irrep  $R$  of  $\mathbb{C}(S_{m+n})$ . More details on the basis in Appendix B.

Here look at  $G_r \equiv G_1 + G_2$ . On gauge invariant operators  $G_r = -G_3 - G_4$ .

We will show

$$\begin{aligned}
\text{tr}(Ad_{G_r}^2) \mathcal{O}_{R_1, R_2; i, j}^R &= Ad_{(G_r)_j^i} Ad_{(G_r)_i^j} \mathcal{O}_{R_1, R_2; i, j}^R = C_2(R) \mathcal{O}_{R_1, R_2; i, j}^R \\
\text{tr}(Ad_{G_1}^2) \mathcal{O}_{R_1, R_2; i, j}^R &= Ad_{(G_1)_j^i} Ad_{(G_1)_i^j} \mathcal{O}_{R_1, R_2; i, j}^R = C_2(R_1) \mathcal{O}_{R_1, R_2; i, j}^R \\
\text{tr}(Ad_{G_2}^2) \mathcal{O}_{R_1, R_2; i, j}^R &= Ad_{(G_2)_j^i} Ad_{(G_2)_i^j} \mathcal{O}_{R_1, R_2; i, j}^R = C_2(R_2) \mathcal{O}_{R_1, R_2; i, j}^R \\
\text{tr}(Ad_{G_1}^2 Ad_{G_2}) \mathcal{O}_{R_1, R_2; i, j}^R &= Ad_{(G_1)_j^i} Ad_{(G_1)_k^j} Ad_{(G_2)_i^k} \mathcal{O}_{R_1, R_2; i, j}^R = C_{112}(R; R_1, R_2, i) \mathcal{O}_{R_1, R_2; i, j}^R \\
\text{tr}(Ad_{G_3}^2 Ad_{G_4}) \mathcal{O}_{R_1, R_2; i, j}^R &= Ad_{(G_3)_j^i} Ad_{(G_3)_k^j} Ad_{(G_4)_i^k} \mathcal{O}_{R_1, R_2; i, j}^R = C_{334}(R; R_1, R_2, j) \mathcal{O}_{R_1, R_2; i, j}^R
\end{aligned} \tag{37}$$

$C_2(R)$  is a Casimir of  $U(N)$  with  $m+n$  boxes.  $C_2(R_1)$  is a Casimir of  $U(N)$  for  $R_1$  with  $m$  boxes.  $C_2(R_2)$  is a Casimir of  $U(N)$  for  $R_2$  with  $n$  boxes.  $C_{112}(R; R_1, R_2, i)$  and  $C_{334}(R; R_1, R_2, j)$  are less familiar. They can be related to elements in  $\mathbb{C}(S_{m+n})$  invariant under  $\mathbb{C}(S_m) \times \mathbb{C}(S_n)$ . They can distinguish between different copies of representations  $(R_1, R_2)$  of the product group in the reduction of a fixed  $R$ . Examples of these eigenvalues will be computed in section 5.

From the point of view of the Casimirs it is clear why the restricted Schur basis is different from the Brauer basis. The Casimir  $\text{tr}(Ad_{G_2} Ad_{G_3})$  appears in the expansion of  $\text{tr}(Ad_{G_B} Ad_{G_B})$  which measures  $\gamma$  of the Brauer basis. The Casimir  $\text{tr}(Ad_{G_1} Ad_{G_2})$  appears in the expansion of  $\text{tr}(Ad_{G_r} Ad_{G_r})$  which measures the  $R$ -label of the restricted Schur basis. It is easy to check that

$$[\text{tr}(Ad_{G_1} Ad_{G_2}), \text{tr}(Ad_{G_2} Ad_{G_3})] = \text{tr}(Ad_{G_1} Ad_{G_2} Ad_{G_3}) - \text{tr}(Ad_{G_1} Ad_{G_3} Ad_{G_2}) \tag{38}$$

Since the two sets of Casimirs do not commute, their eigenstates are different.

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<sup>1</sup> A map, called  $\Sigma$ , from the Brauer algebra  $B_N(m, n)$  to the group algebra of the symmetric group  $S_{m+n}$  was given in [9] and several useful algebraic properties described, and this map was exploited in the construction of the basis in [11]. This map also allows a new holomorphic interpretation of the large  $N$  expansion of 2DYM [21].

### 3.4 Enhanced symmetries and Casimirs : $U(M) \times S_n$ basis for holomorphic functions of complex matrices

Now we are looking at  $M$  complex matrices  $X_1, X_2 \cdots X_M$ . Let us focus on the holomorphic sector. We can consider two different diagonalisations. One based on [10] which keeps  $U(M)$  manifest. One based on [11] which keeps additional  $U(N)$  symmetries manifest. The discussion of the Casimir-diagonal basis connection for the restricted Schur basis proceeds by a straightforward generalisation of section 3.3. We will focus our attention on the basis which keeps  $U(M)$  manifest [10].

The operators are

$$\begin{aligned} \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} &= \frac{1}{n!} \sum_{\alpha} B_{k\beta} D_{ij}^R(\alpha) C_{R, R, i, j}^{\tau, \Lambda, k} \text{tr}(\alpha \mathbf{X}) \\ &= B_{k\beta} C_{R, R, i, j}^{\tau, \Lambda, k} \text{tr}(Q_{ij}^R \mathbf{X}) \end{aligned} \quad (39)$$

where we defined

$$Q_{ij}^R = \frac{1}{n!} \sum_{\alpha} D_{ij}^R(\alpha) \alpha \quad (40)$$

and  $\mathbf{X}$  is defined as  $X_1 \otimes \cdots \otimes X_1 \otimes X_2 \otimes \cdots \otimes X_2 \otimes \cdots \otimes X_M \otimes \cdots \otimes X_M$  with  $\mu_1$  copies of  $X_1$ ,  $\mu_2$  copies of  $X_2$  etc. and  $\mu_1 + \mu_2 + \cdots + \mu_M = n$ . Here the indices  $i, j$  are summed over states in the irrep  $R$  of  $S_n$ , the index  $k$  is summed over states in the irrep  $\mathbf{L}$  of  $S_n$ . The label  $\tau$  runs over the multiplicity of  $\Lambda$  in the tensor product  $R \otimes R$ . The numbers  $C_{R, R, i, j}^{\tau, \Lambda, k}$  are Clebsch-Gordan coefficients. The coefficients  $B_{k\beta}$  are branching coefficients.

The label  $\mathbf{L}$  belongs to  $U(M)$  or by SW duality to  $S_n$ . There are generators  $G_{ab} = \int \text{tr}(X_a \Pi_{X_b})$  which transform the  $a$  indices of  $(X_a)_j^i$ . Quadratic Casimirs measuring  $\mathbf{L}$  are  $Ad_{G_{ab}} Ad_{G_{ba}}$ . The content  $\mu$  is measured by the Cartan elements  $G_{aa}$ . The  $R$  label is related to Casimirs of  $G(\mathcal{L}, X_1) + G(\mathcal{L}, X_2) + \cdots + G(\mathcal{L}, X_M) \equiv G_{\mathcal{L}}$ . Similarly we have  $G_{\mathcal{R}}$  for the right action which also gives the same Casimir. This is essentially because we can think of the  $Q_{ij}^R$  above as  $|R, i\rangle \langle R, j|$ . The  $\beta$  label has to do with the reduction from  $U(M) \rightarrow U(1)^M$  or equivalently by SW duality  $S_n \rightarrow S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_M}$ . In the special case  $M = 2$ , we denote  $X_1, X_2$  as  $X, Y$ . The last equation below is pretty clear from our discussions of the restricted Schurs, where a similar symmetric group reduction multiplicity appears.

We want to show that we can choose bases  $\tau, \beta$  which satisfy

$$\begin{aligned} \text{tr}(Ad_{G_{\mathcal{L}}} Ad_{G_{\mathcal{L}}}) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} &\equiv Ad_{(G_{\mathcal{L}})_j^i} Ad_{(G_{\mathcal{L}})_i^j} \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \\ &= C_2(R) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \\ \text{tr}(Ad_{G_{\mathcal{R}}} Ad_{G_{\mathcal{R}}}) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} &\equiv Ad_{(G_{\mathcal{R}})_j^i} Ad_{(G_{\mathcal{R}})_i^j} \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \\ &= C_2(R) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \\ \text{tr} \otimes \text{tr}(Ad_{G_{\mathcal{R}}} Ad_E Ad_{G_{\mathcal{L}}}) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} &\equiv Ad_{(G_{\mathcal{R}})_j^i} Ad_{E_{lm}^{jk}} Ad_{(G_{\mathcal{R}})_k^m} \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \\ &= C_T(\tau) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \\ \text{tr}(Ad_{G(\mathcal{L}, X)} Ad_{G(\mathcal{L}, X)} Ad_{G(\mathcal{L}, Y)}) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} &\equiv Ad_{(G(\mathcal{L}, X))_j^i} Ad_{(G(\mathcal{L}, X))_k^j} Ad_{(G(\mathcal{L}, Y))_i^k} \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \\ &= C_{XXY}(\beta) \mathcal{O}^{\Lambda, R, \tau, \beta, \mu} \end{aligned} \quad (41)$$

### 3.5 Diagonal bases and Casimirs : the general case of $G \times S_n$

A general construction of diagonal bases of operators compatible with global symmetry group  $G$  using Clebsch-Gordan coefficients of  $G \times S_n$  was given [12]. The operators are of the form  $\mathcal{O}^{\Lambda, \Lambda_1, R, M_\Lambda, \tau_{\Lambda_1 \Lambda}, \tau}$ . In a theory with symmetry group  $G$  there will be corresponding charges  $Q_a$  acting as  $Ad_{Q_a}$  via commutators. Casimirs such as  $Ad_{Q_a} Ad_{Q_a}$ , where the indices are contracted with a Killing form on the Lie algebra, give quadratic Casimirs which measure  $\Lambda$ , which labels irreps. of  $G$ . Higher order invariants give higher Casimirs sensitive to  $\Lambda$ . The  $Q_a$  will include a maximally commuting Cartan sub-algebra which measure the states  $M_\Lambda$  in the irrep.  $\Lambda$ . The  $S_n$  representation  $R$  corresponds to a Young diagram with  $n$  boxes. As in earlier sections there are Casimirs constructed from the  $U(N)$  generators acting on the upper (or lower) indices of the matrix fields, which measure  $R$ . The  $\Lambda_1$  label corresponds to an  $S_n$  which acts simultaneously on the upper and lower  $U(N)$  indices of the matrix fields. Hence the Casimirs constructed from this diagonal  $U(N)$  measure  $\Lambda_1$ . The label  $\tau$  runs over the multiplicity of  $\Lambda_1$  in the Clebsch-Gordan decomposition of the tensor product  $R \otimes R$ . The invariants for can be obtained from  $U(N^2)$  and  $U(N) \times U(N)$  generators, as in sections 3.4 and 4.3. Finally there is the label  $\tau_{\Lambda_1 \Lambda}$  which runs over the multiplicity of the  $G \times S_n$  representation  $V_\Lambda \otimes V_{\Lambda_1}$  in the decomposition of  $V_F^{\otimes n}$ , where  $V_F$  is the representation of  $G$  carried by the fundamental fields. We expect that similar ideas can be used to construct Casimirs which distinguish this multiplicity. To make this more precise we need a better understanding of the commutant  $Com(G \times S_n)$  of  $G \times S_n$  in  $V_F^{\otimes n}$ . A Cartan-like subalgebra will distinguish states in the multiplicity space  $V_{\Lambda, \Lambda_1}$  which form a representation of  $Com(G \times S_n)$ . For the symmetric group a Cartan-like sub-algebra plays a prominent role in the Vershik-Okunkov approach [39] to symmetric group representations. We will use this approach in section 5. A generalisation of these Cartan-like sub-algebras to commutant algebras such as  $Com(G \times S_n)$  and their expression in terms of Noether charges constructed from iterated applications of the generators of  $G$  would lead to the resolution of the parameter  $\tau_{\Lambda, \Lambda_1}$ . In other words it would lead to Casimir-like operators which have the  $\mathcal{O}^{\Lambda, \Lambda_1, R, M_\Lambda, \tau_{\Lambda_1 \Lambda}, \tau}$  as eigenvectors and whose eigenvalues will depend non-trivially on  $\tau_{\Lambda_1 \Lambda}$ . In the case of the full field content of  $\mathcal{N} = 4$  SYM we would want to implement the above programme for  $G = PSU(2, 2|4)$ . It seems likely that this would show up some interesting connections to Yangians and other approaches to integrability in AdS/CFT along the lines of [22].

## 4 Diagonal bases of gauge invariant operators as eigenstates of Casimirs : Proofs

### 4.1 Brauer Basis : One complex matrix

We prove the claims in section 3.2 that the labels on the diagonal basis of operators  $\mathcal{O}_{\alpha,\beta,i,j}^\gamma$  are measured by Casimirs of enhanced symmetries.

#### 4.1.1 The $U(N)$ generators and $C_2(\gamma)$

We will prove the first claim in section 3.2 that Casimirs constructed from  $G_B = G_2 + G_3$  measure the label  $\gamma$ . The action of  $G_B$  is

$$\begin{aligned} [(G_B)_j^i, X_q^p] &= \delta_j^p X_q^i \\ [(G_B)_j^i, (X^*)_q^p] &= -\delta^{ip} (X^*)_q^j \end{aligned} \quad (42)$$

From (13), we see that  $G_B^i_j$  indeed obey relations of  $U(N)$

$$[(G_B)_j^i, (G_B)_l^k] = \delta_j^k (G_B)_l^i - \delta_l^i (G_B)_j^k \quad (43)$$

Defining the field theory commutator  $Ad_{(G_B)_j^i} \equiv [(G_B)_j^i, \cdot]$ , we see that  $Ad_{(G_B)_j^i} Ad_{(G_B)_i^j} \equiv \hat{\mathbf{C}}_2$  acts on the upper indices of  $X \otimes X^*$  the way the quadratic Casimir  $\hat{C}_2 \equiv E_j^i E_i^j$  of  $U(N)$  acts on  $V \otimes \bar{V}$ .

We also have the charges  $\tilde{G}_B = G_1 + G_4$  which act on the fields as

$$\begin{aligned} [(\tilde{G}_B)_j^i, X_q^p] &= -\delta_q^i X_j^p \\ [(\tilde{G}_B)_j^i, (X^*)_q^p] &= \delta_{jq} (X^*)_i^p \end{aligned} \quad (44)$$

The charge  $\tilde{G}_B$  also satisfies the same commutation relation as (43). As long as we act with  $(G_B)$  and  $(\tilde{G}_B)$  on gauge invariant operators, we have  $Ad_{(G_B)} + Ad_{(\tilde{G}_B)} = 0$  because  $Ad_{(G_B)} + Ad_{(\tilde{G}_B)}$  generates the global adjoint gauge transformation  $X \rightarrow UXU^\dagger$ ,  $X^\dagger \rightarrow UX^\dagger U^\dagger$ .

The action of  $\hat{\mathbf{C}}_2 \equiv Ad_{G_B} Ad_{G_B}$  on  $X_{q_1}^{p_1} \cdots X_{q_m}^{p_m} X_{q_{m+1}}^{*p_{m+1}} \cdots X_{q_{m+n}}^{*p_{m+n}}$  leaves fixed the lower indices and acts on the upper indices just the way  $\hat{C}_2$  acts on  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ . Now we know, from Schur-Weyl duality that  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$  decomposes into

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_{\gamma} V_{\gamma}^{U(N)} \otimes V_{\gamma}^{B_N(m,n)} \quad (45)$$

The irrep.  $V_{\gamma}^{B(m,n)}$ , which gives the multiplicity space of  $V_{\gamma}^{U(N)}$ , decomposes into a number of irreps. of  $\mathbb{C}(S_m \times S_n)$ . The branching operators leave the  $V_{\gamma}^{U(N)}$  factor unchanged, and map from one irrep of the  $\mathbb{C}(S_m \times S_n)$  to another. Hence we have

$$Q_{A,ij}^{\gamma} (V^m \otimes \bar{V}^n) = V_{\gamma}^{U(N)} \otimes Q_{A,ij}^{\gamma} (V_{\gamma}^{B_N(m,n)}) \quad (46)$$



This means that the action of the  $U(N)$  generators  $(G_B)$  on  $Q_{A,ij}^\gamma(V^m \otimes \bar{V}^n)$  is the same as on the irrep  $\gamma$  of  $U(N)$ . Hence it follows that

$$\hat{\mathbf{C}}_2 Q_{A,ij}^\gamma(\mathbf{X} \otimes \mathbf{X}^*) = C_2(\gamma) Q_{A,ij}^\gamma(\mathbf{X} \otimes \mathbf{X}^*) \quad (47)$$

and we obtain the first line of (36).

There are higher order Casimirs such as  $\hat{\mathbf{C}}_3 \equiv Ad_{(G_B)_j^i} Ad_{(G_B)_k^j} Ad_{(G_B)_i^k}$

$$\hat{\mathbf{C}}_3 tr_{m,n} (Q_{A,ij}^\gamma(\mathbf{X} \otimes \mathbf{X}^*)) = C_3(\gamma) tr_{m,n} (Q_{A,ij}^\gamma(\mathbf{X} \otimes \mathbf{X}^*)) \quad (48)$$

It is easy to check directly in simple cases, e.g  $m = n = 1$ , that the above equations holds. Consider the case  $m = 1, n = 1$

$$\begin{aligned} \hat{\mathbf{C}}_2 tr X X^\dagger &= 0 \\ \hat{\mathbf{C}}_2 tr X tr X^\dagger &= 2N \left( tr X tr X^\dagger - \frac{1}{N} tr X X^\dagger \right) \end{aligned} \quad (49)$$

Hence the eigenstates of  $\hat{\mathbf{C}}_2$  are  $tr X X^\dagger$  and  $(tr X tr X^\dagger - \frac{1}{N} tr X X^\dagger)$ , which are the simplest examples of operators  $\mathcal{O}_{\alpha,\beta;i,j}^\gamma$ , see (A.37) in [9].

#### 4.1.2 Casimirs $C_2(\alpha)$ and $C_2(\beta)$

Recall that  $G_B$  is a sum

$$G_B = G_2 + G_3 \quad (50)$$

The actions on the fields are as follows

$$\begin{aligned} [(G_2)_j^i, X_q^p] &= \delta_j^p X_q^i \\ [(G_2)_j^i, (X^*)_q^p] &= 0 \\ [(G_3)_j^i, X_q^p] &= 0 \\ [(G_3)_j^i, (X^*)_q^p] &= -\delta^{ip} X_q^j \end{aligned} \quad (51)$$

We can easily check that  $(G_2)_j^i$  and  $(G_3)_j^i$  also obey the commutation relation of  $U(N)$ . Therefore we also have a Casimir which is given by

$$\hat{\mathbf{C}}_2^{++} \equiv Ad_{(G_2)_j^i} Ad_{(G_2)_i^j} \quad (52)$$

Similarly we have another Casimir

$$\hat{\mathbf{C}}_2^{--} \equiv Ad_{(G_3)_j^i} Ad_{(G_3)_i^j} \quad (53)$$

We will prove

$$\hat{\mathbf{C}}_2^{++} tr \left( Q_{(\alpha,\beta),ij}^\gamma(\mathbf{X} \otimes \mathbf{X}^*) \right) = C_2(\alpha) tr \left( Q_{(\alpha,\beta),ij}^\gamma(\mathbf{X} \otimes \mathbf{X}^*) \right)$$

$$\hat{\mathbf{C}}_2^{--} \text{tr} \left( Q_{(\alpha,\beta),ij}^\gamma (\mathbf{X} \otimes \mathbf{X}^*) \right) = C_2(\beta) \text{tr} \left( Q_{(\alpha,\beta),ij}^\gamma (\mathbf{X} \otimes \mathbf{X}^*) \right) \quad (54)$$

The following steps give the result

$$\begin{aligned} & \hat{\mathbf{C}}_2^{++} (Q_{(\alpha,\beta),ij}^\gamma)_J^I (\mathbf{X} \otimes \mathbf{X}^*)_K^J \\ &= (\hat{C}_2^{++})_L^J (Q_{(\alpha,\beta),ij}^\gamma)_J^I (\mathbf{X} \otimes \mathbf{X}^*)_K^L \\ &= (\hat{C}_2^{++} Q_{(\alpha,\beta),ij}^\gamma)_L^I (\mathbf{X} \otimes \mathbf{X}^*)_K^L \\ &= C_2(\alpha) (Q_{(\alpha,\beta),ij}^\gamma)_L^I (\mathbf{X} \otimes \mathbf{X}^*)_K^L \end{aligned} \quad (55)$$

where  $\hat{C}_2^{++} = \sum_{r,s=1}^m \rho_s(E_j^i) \rho_t(E_i^j)$  and  $\hat{C}_2^{--} = \sum_{r,s=m+1}^{m+n} \rho_s(E_j^i) \rho_t(E_i^j)$ .

To get the last equality we can use the expression  $Q_{A,ij}^\gamma = \sum_{m_A} |\gamma \rightarrow A, m_A, i\rangle \langle \gamma \rightarrow A, m_A, j|$  or the expression in terms of the algebra

$$Q_{A,ij}^\gamma = \sum_b B_{\gamma,A,I,i}^\dagger D_{IJ}^\gamma(b) B_{\gamma,A,J,j} b^* \quad (56)$$

This expression is derived in the Appendix B.

#### 4.1.3 $\hat{\mathbf{C}}_{223}$ and $\hat{\mathbf{C}}_{114}$ : Casimir for $i, j$ labels of $Q_{\alpha,\beta;ij}^\gamma$

The commutator action of the generators  $G_2$  and  $G_3$  act on the upper index of  $X$  and  $X^*$  while leaving the lower indices unchanged (see (12) and (14)). When we have a sequence of  $m$  copies of  $X$ 's and  $n$  copies of  $X^*$ , the set of upper indices transforms as  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ , where  $V$  is the fundamental and  $\bar{V}$  the anti-fundamental of  $U(N)$ . The action of  $\hat{\mathbf{C}}_{223} \equiv Ad_{(G_2)_j^i} Ad_{(G_2)_k^j} Ad_{(G_3)_i^k}$  is the same as the action of  $\rho_{\underline{m}}(E_j^i) \rho_{\underline{m}}(E_k^j) \rho_{\underline{n}}(E_i^k)$ , where  $\rho_{\underline{m}}(E)$  denotes the action of  $U(N)$  generators on  $V^{\otimes m}$  and  $\rho_{\underline{n}}(E)$  denotes the action of  $U(N)$  generators on  $\bar{V}^{\otimes n}$ .

$$\begin{aligned} \hat{C}_{++-} &= \sum_{r,s=1}^m \sum_{t=m+1}^n \rho_r(E_j^i) \rho_s(E_k^j) \rho_t(E_i^k) \\ &= -N \sum_{r,t} \left( 1 + \frac{1}{N} \sum_{s \neq r} (rs) \right) C_{r\bar{t}} \end{aligned} \quad (57)$$

This will be shown in the appendix A.2.

The action  $\hat{\mathbf{C}}_{223}$  on the operators is

$$\begin{aligned} & \hat{\mathbf{C}}_{223} (Q_{A,ij}^\gamma)_I^J (\mathbf{X} \otimes \mathbf{X}^*)_J^I \\ &= (Q_{A,ij}^\gamma)_I^J (\hat{C}_{++-})_K^I (\mathbf{X} \otimes \mathbf{X}^*)_J^K \\ &= (\hat{C}_{++-} Q_{A,ij}^\gamma)_I^J (\mathbf{X} \otimes \mathbf{X}^*)_J^K \\ &= (\hat{C}_{++-} |\gamma, A, m_A, i\rangle \langle \gamma, A, m_A, j|)_I^J (\mathbf{X} \otimes \mathbf{X}^*)_J^K \\ &= C_{++-}(\gamma, A, i) (Q_{A,ij}^\gamma)_I^J (\mathbf{X} \otimes \mathbf{X}^*)_J^K \end{aligned} \quad (58)$$

Note that  $Ad_{G_1}$  and  $Ad_{G_4}$  act on the lower indices of  $X, X^*$ . To measure  $j$  we consider

$$\hat{\mathbf{C}}_{114} (Q_{A,ij}^\gamma)_I^J (\mathbf{X} \otimes \mathbf{X}^*)_J^I$$

$$\begin{aligned}
&= (Q_{A,ij}^\gamma)_I^J (\hat{C}_{++-})_J^K (\mathbf{X} \otimes \mathbf{X}^*)_K^I \\
&= (Q_{A,ij}^\gamma \hat{C}_{++-})_I^K (\mathbf{X} \otimes \mathbf{X}^*)_K^I \\
&= C_{++-}(\gamma, A, j) (Q_{A,ij}^\gamma)_I^K (\mathbf{X} \otimes \mathbf{X}^*)_K^I
\end{aligned} \tag{59}$$

In section 5, we show explicitly how to find the basis of states which diagonalise the Casimir  $\hat{C}_{++-}$  and the eigenvalues  $C_{++-}(\gamma, A, i)$  in a simple example.

## 4.2 Casimirs and restricted Schur Basis for $X, X^\dagger$

Instead of using the Brauer algebra  $B_N(m, n)$  and its reduction to the sub-algebra  $\mathbb{C}(S_m) \times \mathbb{C}(S_n)$  we are now using the algebra  $\mathbb{C}(S_{m+n})$  and its reduction to  $\mathbb{C}(S_m) \times \mathbb{C}(S_n)$ . The Brauer algebra is the commutant of  $U(N)$  acting on  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ . This representation is equivalent to the action of  $G_2, G_3$  (see (14)) on the upper indices of  $X, X^*$ . The lower indices are similarly acted on by  $G_1, G_3$ . On the other hand,  $\mathbb{C}(S_{m+n})$  is dual to the action of  $U(N)$  on  $V^{\otimes(m+n)}$ . This is equivalent to the action of  $G_1, G_2$  (see (14)) on the upper indices of  $X, X^\dagger$ . The lower indices are similarly acted on by  $G_3, G_4$ . The space of gauge invariant operators can be constructed as  $tr_{m,n}(b(\mathbf{X} \otimes \mathbf{X}^*))$  or as  $tr_{m+n}(\sigma(\mathbf{X} \otimes \mathbf{X}^\dagger))$ : the two constructions can be related via the invertible map  $\Sigma$  from Brauer algebra to symmetric group, but give distinct bases diagonalising the two-point functions. The calculations are closely parallel, with the role of the pair  $(G_2, G_3)$  of section 4.1 being now played by the pair  $(G_1, G_2)$ , and the pair  $(G_1, G_4)$  of section 4.1 now played by  $(G_3, G_4)$ .

$tr(Ad_{G_r} Ad_{G_r})$  acts on the upper indices of  $m$  copies on  $X$  and  $n$  copies of  $X^\dagger$  as the Casimir of  $U(N)$  acting on  $V^{\otimes(m+n)}$ . This Casimir action can be expressed in terms of symmetric group action (a step which is useful in the string theory of two dimensional Yang Mills [23])

$$\begin{aligned}
\hat{C}_2 &= \sum_{r,s} \rho_r(E_j^i) \rho_s(E_i^j) \\
&= (m+n)N + \sum_{r \neq s} (rs)
\end{aligned} \tag{60}$$

The derivation of this is in the Appendix A. This relation between the  $U(N)$  group and the symmetric group follows essentially from Schur-Weyl duality

$$V^{\otimes m+n} = \bigoplus_R V_R^{U(N)} \otimes V_R^{S_{m+n}} \tag{61}$$

Using the analogous steps to section 4.1.1 we see

$$\hat{C}_2(G_r) (Q_{R_1, R_2; ij}^R)_I^J (\mathbf{X} \otimes \mathbf{X}^\dagger)_J^I = C_2(R) (Q_{R_1, R_2; ij}^R)_I^J (\mathbf{X} \otimes \mathbf{X}^\dagger)_J^I \tag{62}$$

hence proving the first line of (37)

Now we turn to the measurement of the quantum numbers  $R_1, R_2$  in the restricted Schur basis. The operator  $tr(Ad_{G_1} Ad_{G_1})$  acts like the Casimir  $\rho_{\underline{n}}(E_j^i) \rho_{\underline{n}}(E_i^j)$ . The  $\rho_{\underline{n}}(E_j^i)$  denotes

the action of  $U(N)$  on the  $n$  fold tensor product of the fundamental. This can be expressed as  $Nn + \sum_{r \neq s=1}^n (rs)$  in the group algebra of  $S_n$ . As in section 4.1.2 the action of the  $\mathbb{C}(S_n)$  element on  $Q_{R_1, R_2; i, j}^R$  results in  $C_2(R_2)$ . Likewise  $tr(Ad_{G_2} Ad_{G_2})$  gives  $C_2(R_1)$ .

The operator  $tr Ad_{G_1} Ad_{G_1} Ad_{G_2}$  acts on the upper indices of  $X_{j_1}^{i_1} \cdots X_{j_m}^{i_m} X_{j_{m+1}}^{i_{m+1}} \cdots X_{j_{m+n}}^{i_{m+n}}$  as  $\rho_{\underline{n}}(E_j^i) \rho_{\underline{n}}(E_k^j) \rho_{\underline{m}}(E_i^k) \equiv \hat{C}_{112}$  acting on  $V^{\otimes m} \otimes V^{\otimes n}$ .

$$\begin{aligned} \hat{C}_{112} &= \sum_{r=m+1}^{m+n} \sum_{s=m+1}^{m+n} \sum_{t=1}^m \rho_r(E_j^i) \rho_s(E_k^j) \rho_t(E_i^k) \\ &= N \sum_{r,t} \left( 1 + \frac{1}{N} \sum_{s \neq r} (rs) \right) (rt) \end{aligned} \quad (63)$$

The element  $\sum_{r=m+1}^{m+n} \sum_{s=m+1}^{m+n} \sum_{t=1}^m (rs)(rt)$  is in  $\mathbb{C}(S_{m+n})$  but is invariant under the subalgebra  $\mathbb{C}(S_m) \times \mathbb{C}(S_n)$ . It has matrix elements

$$\langle R \rightarrow R_1, R_2; m'_{R_1}, m'_{R_2}, i' | \hat{C}^{112} | R \rightarrow R_1, R_2, m_{R_1}, m_{R_2}, i \rangle = (C_{112})_{i'i} \delta_{m_{R_1}, m'_{R_1}} \delta_{m_{R_2}, m'_{R_2}} \quad (64)$$

By diagonalising the matrix  $(C_{112})_{ii'}$  we can find the eigenstates and eigenvalues, which distinguish different indices  $i$ . An example is given in section 5.

### 4.3 $U(M) \times S_n$ basis : for holomorphic $X_1, X_2, \dots, X_M$

#### 4.3.1 The $\tau$ multiplicity and the $U(N^2)$ symmetry

The  $\tau$  label runs over the multiplicity of  $\Lambda$  in the Clebsch-Gordan decomposition of  $R \otimes R$ . Another way to say this is that it runs over the multiplicity of the irrep  $\Lambda$  of the diagonal subgroup  $S_n$  in the irrep  $R \otimes R$  of  $S_n \times S_n$ . The upper and lower indices of the  $X$ 's in equation (39) have been projected to  $R$  due to the insertion of  $Q_{ij}^R$ . We can think of  $Q_{ij}^R$  as  $|R, i\rangle \langle R, j|$ . The product group  $S_n \times S_n$  is SW dual to  $U(N) \times U(N)$ . where the two  $U(N)$  act on upper and lower indices respectively. The action on upper and lower indices is generated by  $G_{\mathcal{L}}$  and  $G_{\mathcal{R}}$  defined in section 3.4. The operators in equation (39) are linear combinations of

$$(X_{a_1})_{q_1}^{p_1} (X_{a_2})_{q_2}^{p_2} \cdots (X_{a_n})_{q_n}^{p_n} \quad (65)$$

To be more precise we take  $a_1, \dots, a_{\mu_1} = 1, a_{\mu_1+1}, \dots, a_{\mu_2} = 2$  etc. The symmetry generator of interest  $G_{\mathcal{L}}$  acts in the same way irrespective of what value the  $a_i$  have. The diagonal  $S_n$  is SW dual to  $U(N^2)$ . By the general theorem discussed in the introduction, this means that the  $\tau$  multiplicity is related to the reduction of  $U(N^2)$  to  $U(N) \times U(N)$ . The  $U(N^2)$  has been discussed in section 2.2.

Consider therefore the operator  $Ad_{E_{lm}^{jk}} Ad_{(G_{\mathcal{L}})_k^m} Ad_{(G_{\mathcal{R}})_j^l}$  on the above. This is made from the  $U(N) \times U(N)$  along with the  $U(N^2)$  and is invariant under the global gauge symmetry since it has all indices contracted. In other words we are considering

$$[E_{lm}^{jk}, [(G_{\mathcal{L}})_k^m, [(G_{\mathcal{R}})_j^l, \mathbf{X}]]] \quad (66)$$

The adjoint action of each symmetry generator on  $\mathbf{X}$  produces a sum of  $n$  terms, each following from the linear transformation of different factors in the  $n$ -fold product. The actions on a single  $X$  is given by

$$\begin{aligned} Ad_{E_{lm}^{jk}}(X_q^p) &= [E_{lm}^{jk}, X_q^p] = \delta_m^p \delta_q^j X_l^k \\ Ad_{(G_{\mathcal{L}})_j^l}(X_q^p) &= [(G_{\mathcal{L}})_j^l, X_q^p] = \delta_j^p X_q^l \\ A_{(G_{\mathcal{R}})_k^m}(X_q^p) &= [(G_{\mathcal{R}})_k^m, X_q^p] = \delta_q^m X_k^p \end{aligned} \quad (67)$$

For any of the symmetry generators  $Ad_S(\mathbf{X})$  can be expanded as

$$Ad_S = \sum_{i=1}^n Ad_S^{(i)} \quad (68)$$

where  $Ad_S^{(i)}$  acts on the  $i$ 'th factor of the  $n$ -fold product.

The action of the operator can be expressed in terms of elements in the symmetric group

$$\begin{aligned} & Ad_{E_{lm}^{jk}} Ad_{(G_{\mathcal{L}})_k^m} Ad_{(G_{\mathcal{R}})_j^l} - N Ad_{(G_{\mathcal{L}})_k^m} A_{(G_{\mathcal{L}})_m^k} - N Ad_{(G_{\mathcal{R}})_j^l} A_{(G_{\mathcal{R}})_l^j} - Ad_{E_{lm}^{jk}} Ad_{E_{jk}^{lm}} - 2N^2 n \\ &= \sum_{r \neq s \neq t} (rs)_{\mathbf{u}}(rt)_{\mathbf{l}} \end{aligned} \quad (69)$$

The derivation of this is given in the Appendix A.4. We next show that the above field theory operator can measure the multiplicity.

### 4.3.2 Proof

In [10] the diagonal basis contains  $\tau$  which runs over the multiplicity of the irrep.  $\Lambda$  of  $S_n$  in the tensor product decomposition  $R \otimes R$  of the irrep.  $R$  of  $S_n$ .

The Casimirs  $C_T(\tau)$  are defined generally for tensor products of  $S_n$  representations. For tensor product states  $|R_1, R_2, m_1, m_2\rangle$  we have the action of  $\hat{C}_T \equiv \sum_{r \neq s \neq t} (rs) \otimes (rt)$  as

$$\begin{aligned} \hat{C}_T |R_1, R_2, m_1, m_2\rangle &= \sum_{r \neq s \neq t} (rs) \otimes (rt) |R_1, R_2, m_1, m_2\rangle \\ &= \sum_{r \neq s \neq t} D_{n_1 m_1}^{R_1}((rs)) D_{n_2 m_2}^{R_2}((rt)) |R_1, R_2, n_1, n_2\rangle \end{aligned} \quad (70)$$

The tensor product space can be decomposed in terms of irreps. of  $S_n$  using the Clebsch-Gordan coefficients :

$$|R_1, R_2, m_1, m_2\rangle = C_{R_1, R_2, m_1, m_2}^{\Lambda, m_{\Lambda}, \tau} |\Lambda, m_{\Lambda}, \tau\rangle \quad (71)$$

For the application at hand we will be interested in the special case  $R_1 = R_2 = R$ . The element  $\hat{C}_T$  is an element of the group algebra  $\mathbb{C}(S_n) \times \mathbb{C}(S_n)$  which is invariant under the diagonal  $\mathbb{C}(S_n)$ . The Clebsch-Gordan decomposition is equivalent to the reduction from the

tensor product algebra to its diagonal subalgebra. Since  $\hat{C}_T$  commutes with the diagonal  $\mathbb{C}(S_n)$ , by Schur's Lemma, its matrix elements in the  $S_n$  basis satisfy :

$$\langle \Lambda, m_\Lambda, \tau | \hat{C}_T | \Lambda', m'_{\Lambda'}, \tau' \rangle = \delta_{\Lambda\Lambda'} \delta_{m_\Lambda m'_{\Lambda'}} (C_T)_{\tau\tau'} \quad (72)$$

By choosing a basis in the multiplicity space such that  $(C_T)_{\tau\tau'}$  is diagonalised, we have

$$\langle \Lambda, m_\Lambda, \tau | \hat{C}_T | \Lambda', m'_{\Lambda'}, \tau' \rangle = \delta_{\Lambda\Lambda'} \delta_{m_\Lambda m'_{\Lambda'}} \delta_{\tau\tau'} C_T(\tau) \quad (73)$$

Inserting a complete set of states in the left of (70) and taking an overlap with a state labelled by irreps of the diagonal  $S_n$  :

$$\begin{aligned} & \langle \Lambda, m_\Lambda, \tau | \hat{C}_T | \Lambda', m'_{\Lambda'}, \tau' \rangle \langle \Lambda', m'_{\Lambda'}, \tau' | R_1, R_2, m_1, m_2 \rangle \\ &= C_T(\tau) \langle \Lambda, m_\Lambda, \tau | R_1, R_2, m_1, m_2 \rangle \\ &= C_{R_1, R_2, m_1, m_2}^{\Lambda, m_\Lambda, \tau} C_T(\tau) \end{aligned} \quad (74)$$

Taking the same matrix element on the right of (70) gives the identity

$$C_{R_1, R_2, n_1, n_2}^{\Lambda, m_\Lambda, \tau} C_T(\tau) = \sum_{r \neq s \neq t} D_{n_1 m_1}^{R_1}((rs)) D_{n_2 m_2}^{R_2}((rt)) C_{R, R, m_1, m_2}^{\Lambda, m_\Lambda, \tau} \quad (75)$$

Now consider the field theory operators. In (69), we expressed the action by commutators of certain sequences of enhanced symmetries in terms of generalised Casimirs such as the above. We wrote it in terms of  $\sum_{r,s,t} (rs)_u (rt)_l$  indicating the action on the upper and lower indices. We find

$$\begin{aligned} & \sum_{r,s,t} (rs)_u (rt)_l \sum_{\alpha} B_{k\beta} D_{ij}^R(\alpha) C_{R,R,i,j}^{\tau, \Lambda, k} (X_{a_1})_{i_{\alpha(1)}}^{i_1} \cdots (X_{a_n})_{i_{\alpha(n)}}^{i_n} \\ & \sum_{r,s,t} \sum_{\alpha} B_{k\beta} D_{ij}^R((rs)\alpha(rt)) C_{R,R,i,j}^{\tau, \Lambda, k} (X_{a_1})_{i_{\alpha(1)}}^{i_1} \cdots X_{a_n i_{\alpha(n)}}^{i_n} \end{aligned} \quad (76)$$

The action of  $(rs)$  on the upper indices and  $(rt)$  on the lower indices of the  $X$ 's have been absorbed into a re-definition of  $\alpha$ . Expanding

$$\begin{aligned} & \sum_{r,s,t} \sum_{\alpha} B_{k\beta} D_{ik}^R((rs)) D_{kl}^R(\alpha) D_{lj}^R((rt)) C_{R,R,i,j}^{\tau, \Lambda, k} (X_{a_1})_{i_{\alpha(1)}}^{i_1} \cdots (X_{a_n})_{i_{\alpha(n)}}^{i_n} \\ &= C_T(\tau) \sum_{\alpha} B_{k\beta} C_{RRkl}^{\Lambda, m_\Lambda, \tau} D_{kl}^R(\alpha) (X_{a_1})_{i_{\alpha(1)}}^{i_1} \cdots (X_{a_n})_{i_{\alpha(n)}}^{i_n} \\ &= C_T(\tau) \mathcal{O}_{R, \beta, \mu, \tau}^{\Lambda, R, \beta, \mu, \tau} \end{aligned} \quad (77)$$

In the second line we used (75). We have thus proved the third line of (41)

## 5 Examples of generalised Casimirs for resolving multiplicity

In this section, we explicitly calculate the action of the generalised Casimirs for some examples. We use the fact that the generalised Casimirs ( $u(N)$  invariants) can be expressed in terms of the symmetric group or the Brauer algebra, which is explained in the Appendix A.

As we see in (64) or (72) (or more generally (119)), the actions of the Casimirs can be effectively expressed by matrices whose sizes are given by the multiplicities. We explicitly obtain these matrices for some examples.

Calculations in this section have heavily used the SAGE <sup>2</sup> interface to SYMMETRICA<sup>3</sup> to get orthogonal matrix representations of the symmetric group and the MAXIMA <sup>4</sup> for diagonalisation of matrices.

## 5.1 One complex matrix : Brauer

In this subsection, we treat the Brauer basis of non-holomorphic operators constructed from one complex matrix, which is explained in section 3.2 and appendix B. This basis is provided by the operator  $Q_{A,ij}^\gamma$ . We first construct a representation basis  $|\gamma, A, m_A, i\rangle$  representing the reduction of  $\gamma$  to  $A$ , where  $m_A$  labels states in  $A$  and  $i$  labels the multiplicity of this reduction. We next express the generalised Casimir (57) as a matrix whose size is given by the multiplicity.

We consider the representation  $\gamma = (k = 2, \gamma^+ = [1], \gamma^- = [1])$  of  $B_N(3, 3)$ . This is the simplest case where the nontrivial multiplicity occurs. In this case the representation  $A = ([2, 1], [2, 1])$  of the subalgebra  $\mathbb{C}(S_3 \times S_3)$  appears with the multiplicity 2 due to  $M_A^\gamma = 2$  (see (3.13) in [9] for the definition of  $M_A^\gamma$ ). Because the dimension of the  $A$  is 4, there are 8 states labelled by the  $\gamma$  and the  $A$ . To identify these states, we use Cartan elements for  $\mathbb{C}(S_3 \times S_3)$ , which are given by  $X_2^+ = (12)$ ,  $X_3^+ = (13) + (23)$ ,  $X_2^- = (\bar{1}\bar{2})$  and  $X_3^- = (\bar{1}\bar{3}) + (\bar{2}\bar{3})$ . As is reviewed in the Appendix C, four sets of eigenvalues of Cartan elements  $(X_2^+, X_3^+, X_2^-, X_3^-) = (1, -1, 1, -1)$ ,  $(1, -1, -1, 1)$ ,  $(-1, 1, 1, -1)$ ,  $(-1, 1, -1, 1)$  correspond to  $m_A = 1, 2, 3, 4$ .

Representation matrices for  $\gamma = (k = 2, \gamma^+ = [1], \gamma^- = [1])$  are constructed in the Appendix D, and we use them. For example, two orthonormalised states with eigenvalues  $(D^\gamma(X_2^+), D^\gamma(X_3^+), D^\gamma(X_2^-), D^\gamma(X_3^-)) = (1, -1, 1, -1)$  can be found as

$$\begin{aligned} v_1 &= \frac{1}{2\sqrt{5}} (e_1 + e_3 - e_5 - e_6 + e_7 + e_9 - e_{11} - e_{12} - e_{13} - e_{14} - e_{15} - e_{16} + e_{17} + e_{18}) \\ v_2 &= \frac{1}{6\sqrt{5}} (4e_1 - 5e_2 + 4e_3 - 5e_4 - e_5 - e_6 + 4e_7 - 5e_8 + 4e_9 - 5e_{10} \\ &\quad + e_{11} + e_{12} + e_{13} + e_{14} + e_{15} + e_{16} - 2e_{17} - 2e_{18}) \end{aligned} \tag{78}$$

where  $e_i$  ( $i = 1, \dots, 18$ ) are basis vectors of the representation  $\gamma$  of  $B_N(3, 3)$ . These two states can be written as  $|\gamma, A, (1, -1, 1, -1), i\rangle$  ( $i = 1, 2$ ).

These two states should have different multiplicity labels, and we will see that they are resolved by eigenvalues of the generalised Casimir operator introduced in (57).

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<sup>2</sup> <http://www.sagemath.org/>

<sup>3</sup> [http://www.neu.uni-bayreuth.de/de/Uni\\_Bayreuth/Fakultaeten/1\\_Mathematik\\_Physik\\_und\\_Informatik/Fachgruppe\\_Informatik/prof\\_diskrete\\_algorithmen/en/research/SYMMETRICA/index.html](http://www.neu.uni-bayreuth.de/de/Uni_Bayreuth/Fakultaeten/1_Mathematik_Physik_und_Informatik/Fachgruppe_Informatik/prof_diskrete_algorithmen/en/research/SYMMETRICA/index.html)

<sup>4</sup> <http://maxima.sourceforge.net/>

$$T^{++-} \equiv \sum_{r \neq s=1}^3 \sum_{t=4}^6 (rs) C_{r\bar{t}} \quad (79)$$

Action of this operator on the two states are given by

$$\frac{1}{5} \begin{pmatrix} 4N + 12 & -3N + 6 \\ -3N + 6 & -4N + 3 \end{pmatrix} \quad (80)$$

where  $(ij)$  component represents  $\langle v_i | T | v_j \rangle$ . The eigenvalues are  $-(3 \pm \sqrt{4N^2 + 9})/2$ . In this way, the action of the Casimir on the representation basis can be expressed by the  $2 \times 2$  matrix.

We also obtained another set of eigenvectors with eigenvalues  $(1, -1, -1, 1)$  corresponding to another state of  $A = ([2, 1], [2, 1])$ . We found these vectors are also labelled by the same eigenvalues of  $T^{++-}$ . This fact is what is expected from the Schur's lemma. We can check that the Casimir (79) commutes with all elements in the subalgebra  $\mathbb{C}(S_3 \times S_3)$ . Therefore the Casimir acts diagonally in the state labels  $m_\Lambda$  as in (72).

## 5.2 One Complex Matrix : Restricted Schur

In this subsection, another basis of one complex model, restricted schur basis, is investigated. This basis is given by  $Q_{R_1 R_2; ij}^R$ .

We consider the case  $S_6$  reducing to  $S_3 \times S_3$  as a example. In the case, we have the multiplicity is 2 for  $R_1 = [2, 1], R_2 = [2, 1]$  and  $R = [3, 2, 1]$  because the multiplicity is given by the Littlewood-Richardson coefficient,  $g([2, 1], [2, 1]; [3, 2, 1]) = 2$ . There are 8 states labelled by the  $R_1$  and  $R_2$  inside the representation  $R$ . To look for the states, we use Cartan elements for  $\mathbb{C}(S_3 \times S_3)$ , which are denoted by  $(X_2^L, X_3^L, X_2^R, X_3^R)$ . These 8 states have eigenvalues  $(X_2^L, X_3^L) = (\pm 1, \mp 1)$  and  $(X_2^R, X_3^R) = (\pm 1, \mp 1)$ . Representation matrices for  $R = [3, 2, 1]$  can be obtained using the SYMMETRICA<sup>5</sup>. The dimension of this representation is 16. Let  $e_i$  ( $i = 1, \dots, 16$ ) be basis vectors of the representation matrices.

Two orthonormalised eigenvectors with  $(D^R(X_2^L), D^R(X_3^L), D^R(X_2^R), D^R(X_3^R)) = (1, -1, 1, -1)$  are found to be

$$\begin{aligned} & \frac{\sqrt{2}}{4} \left( e_3 + \frac{1}{\sqrt{3}} e_5 - \sqrt{5} e_{13} - \sqrt{\frac{5}{3}} e_{15} \right) \\ & \frac{\sqrt{10}}{8} \left( e_3 + \frac{1}{\sqrt{3}} e_5 - \sqrt{3} e_8 - \frac{3}{\sqrt{5}} e_{10} - \frac{1}{\sqrt{5}} e_{13} + \frac{1}{\sqrt{15}} e_{15} \right) \end{aligned} \quad (81)$$

These should be written as  $|R, R_1, R_2, (1, -1, 1, -1), i\rangle$ , where  $i$  takes 1, 2. The action of the Casimir  $\sum_{s \neq r=1}^3 \sum_{t=4}^6 (rs)(rt)$  on these states are expressed by the following  $2 \times 2$  matrix

$$\begin{pmatrix} -4 & \sqrt{5} \\ \sqrt{5} & 1 \end{pmatrix} \quad (82)$$

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<sup>5</sup> We use the function `symmetrca.odg` in SYMMETRICA which gives the symmetric group matrix elements in an orthonormal bases.



with eigenvalues  $(3 \pm 3\sqrt{5})/2$ .

We also obtained another set of eigenvectors with eigenvalues  $(1, -1, -1, 1)$ , and checked these are labelled by the same eigenvalues of the Casimir. Because the Casimir commutes with all elements of  $S_3 \times S_3$ , it is consistent with the Schur's lemma.

### 5.3 Two complex matrices : $U(2)$ covariant

In this subsection, we study the  $\tau$  multiplicity which arises in holomorphic operators constructed from two complex matrices, which is explained in section 4.3. This multiplicity is related to the reduction  $S_n \times S_n \rightarrow S_n$ .

We consider the case of  $n = 5$ , which is the simplest case to see the multiplicity. The (inner) tensor product between  $[3, 1, 1]$  and itself can be decomposed into

$$[3, 1, 1] \otimes [3, 1, 1] = [5] \oplus [4, 1] \oplus 2[3, 2] \oplus [3, 1, 1] \oplus 2[2, 2, 1] \oplus [2, 1, 1, 1] \oplus [1, 1, 1, 1, 1] \quad (83)$$

Here two irreducible representations  $[3, 2]$  and  $[2, 2, 1]$  appear with the multiplicity 2. From representation matrices for the representation of  $[3, 1, 1]$  whose dimension is 6, we construct  $36 \times 36$  matrices by taking the (inner) tensor product of them. To identify states corresponding to  $[3, 2]$  and  $[2, 2, 1]$ , we use Cartan elements for  $\mathbb{C}(S_5)$ , which are denoted by  $X_i$  ( $i = 2, \dots, 5$ ). Five components in the representation  $[3, 2]$  can be labelled by eigenvalues of the Cartan elements as  $(R^{R \otimes R}(X_2), R^{R \otimes R}(X_3), R^{R \otimes R}(X_4), R^{R \otimes R}(X_5)) = (-1, 1, 0, 2), (1, -1, 0, 2), (-1, 1, 2, 0), (1, -1, 2, 0), (1, 2, -1, 0)$ . These correspond to the five standard tableaux listed in table 3. On the other hand, five states in the representation  $[2, 2, 1]$  are by  $(-1, -2, 1, 0), (-1, 1, -2, 0), (1, -1, -2, 0), (-1, 1, 0, -2), (1, -1, 0, -2)$ . See figure 4 for standard tableaux corresponding to these sets of eigenvalues.

Eigenvectors with  $(1, 2, -1, 0)$ , which corresponds to a state of  $[3, 2]$ , can be found as

$$\begin{aligned} u_1 &= \frac{1}{2\sqrt{3}} (2e_1 - e_8 - e_{15} - e_{22} - e_{29} + 2e_{36}) \\ u_2 &= \frac{1}{6\sqrt{2}} \left( -2e_1 + e_8 + \sqrt{15}e_{10} + e_{15} + \sqrt{15}e_{17} + \sqrt{15}e_{20} - e_{22} + \sqrt{15}e_{27} - e_{29} + 2e_{36} \right) \end{aligned}$$

The action of the Casimir  $\sum_{r \neq s \neq t} (rs) \otimes (rt)$  on these states can be expressed, using (70), by the following matrix

$$\begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} \quad (84)$$

We also computed eigenvectors with  $(1, -1, 2, 0)$  corresponding to another state in the representation  $[3, 2]$ , which gave the same eigenvalues as the above. This is consistent with the Schur's lemma because the Casimir is invariant under the diagonal  $S_5$ .

As for the representation  $[2, 2, 1]$ , we obtained orthonormalised two eigenvectors with eigenvalues  $(1, -1, 0, -2)$  as

$$v_1 = \frac{1}{2\sqrt{33}} \left( \sqrt{10}e_2 + \sqrt{10}e_7 - \sqrt{5}e_8 + \sqrt{5}e_{15} - 2\sqrt{6}e_{19} - 2\sqrt{3}e_{20} \right)$$

$$\begin{aligned}
& +\sqrt{5}e_{22}, +2\sqrt{3}e_{27} - \sqrt{5}e_{29} - \sqrt{10}e_{30} - 2\sqrt{6}e_{33} - \sqrt{10}e_{35} \Big) \\
v_2 = & \frac{\sqrt{11}}{44\sqrt{6}} \Big( -\sqrt{30}e_2 + 11\sqrt{2}e_4 - \sqrt{30}e_7 + \sqrt{15}e_8 + 11e_{10} - \sqrt{15}e_{15} \\
& -11e_{17} + 11\sqrt{2}e_{18} - 5\sqrt{2}e_{19} - 5e_{20} - \sqrt{15}e_{22} + 5e_{27} \\
& +\sqrt{15}e_{29} + \sqrt{30}e_{30} - 5\sqrt{2}e_{33} + \sqrt{30}e_{35} \Big)
\end{aligned} \tag{85}$$

In this case, the action of the Casimir is given by

$$\frac{2}{11} \begin{pmatrix} -18 & 10\sqrt{6} \\ 10\sqrt{6} & 7 \end{pmatrix} \tag{86}$$

whose eigenvalues are  $-6$  and  $4$ . We also obtained the same eigenvalues for eigenvectors with eigenvalues  $(1, -1, -2, 0)$  which is another states of  $[2, 2, 1]$ .

## 6 Discussion

The enhanced symmetries are broken at non-zero coupling. The breaking should lead to Ward identities which control the mixing between the free field diagonal basis labels caused by the action of the one and higher loop Hamiltonians. The mixing has been studied explicitly in the 1-loop case [24], where it was shown that the mixing of the  $R$ -label was limited by a simple rule of adding and removing a box of the Young diagram.

Often the same sector of the theory admits different diagonal bases. The case of one complex matrix has been discussed at length. Another example in  $\mathcal{N} = 4$  SYM is the holomorphic sector of 3 complex matrices. We can diagonalise with the  $U(3) \times S_n$  covariant method of [12], where the  $S_n$  is dual to  $U(N^2)$ . Alternatively we have the method of [11] which keeps  $S_{n_1} \times S_{n_2} \times S_{n_3}$  manifest. The latter is Schur-Weyl dual to  $U(N) \times U(N) \times U(N)$ . The transformation between the two bases should have a group theoretic meaning in terms of the relations between the different unitary groups involved.

### 6.1 An extremality property

In the context of the holomorphic sector of a single matrix  $X$  and the related half-BPS LLM geometries, we can view the higher Casimirs as space-time charges encoded in the asymptotics of the gravitational fields [15]. It is reasonable to assume that the existence of these charges holds beyond the SUSY case even though the available description uses the multipoles of the  $u$ -function characterising the LLM geometries. In the case of one complex matrix, it was argued [9] that the Brauer basis has an interpretation in terms of branes and anti-branes. Recall that the label  $\gamma$  is equivalent to  $(\gamma_+, \gamma_-, k)$  where  $\gamma_+$  is a partition of  $m - k$  and  $\gamma_-$  is a partition of  $n - k$ , expressed as  $\gamma_+ \vdash m - k, \gamma_- \vdash n - k$ . In particular it was argued that  $k = 0$  operators correspond to the ground state of a brane corresponding to  $R \vdash m, S \vdash n$ . The higher  $k$  operators are excited states. In support of this interpretation

we now see that the  $k = 0$  operators obey a type of extremality condition. For fixed  $C_2(\gamma)$ , as we increase  $k$ , the energy increases. So the  $k = 0$  states are minimal energy states for fixed higher charges.

## 6.2 Reduced quantum mechanics

Much of our discussion can be carried over to the case of the reduced quantum mechanics coming from the complex matrices. As an example consider the case of a single complex matrix. In the reduced quantum mechanics, we can use our  $X, X^\dagger$  operators by replacing  $X \rightarrow A^\dagger$  and  $Y \rightarrow B^\dagger$ . The main point that different diagonal bases are related to different enhanced symmetries can be expressed in the reduced models.

## 6.3 Finite $N$

The discussion of the Brauer algebra construction of gauge invariant operators has some interesting subtleties at finite  $N$ . Two things happen at finite  $N$ . The Brauer algebra  $B_N(m, n)$  maps onto the commutant of  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$  but the map is not one-to-one. This is a feature that is seen already in the case of the duality between symmetric groups and the unitary group action on  $V^{\otimes m}$ . Unlike the symmetric group case, the Brauer algebra has structure constants that themselves depend on  $N$ . It ceases to be semi-simple in the case of  $m + n > N$ . This means there is no longer a non-degenerate trace on the algebra, which implies that the general construction of matrix units in the Appendix does not work. A related consequence is that some of the projectors constructed explicitly in [9] become singular when  $m + n > N$ . The systematic treatment of these singularities for the general  $Q_{A,i,j}^\gamma$  is a problem we leave for the future. In some special cases it is clear what the effects of finite  $N$  are. Indeed for  $k = 0$ , where the  $Q_{A,i,j}^\gamma$  are ordinary projectors  $P_{R\bar{S}}$  the cutoff is simply  $c_1(R) + c_1(S) \leq N$ , which was interpreted as an exclusion principle for brane-anti-brane ground states [9]. The finite  $N$  effects in the restricted Schur description of the gauge invariant operators in the  $X, X^\dagger$  sector are easier to handle [11] since we do not have to deal with the non-semisimplicity issue. Given that the same space of operators is being described by  $B_N(m, n)$  and  $S_{m+n}$ , we can expect to find some information about finite  $N$  effects on the Brauer algebra using this set-up. For some mathematical work on finite  $N$  effects in algebra  $B_N(m, n)$  see [25]. An interesting future direction is to give a complete finite  $N$  discussion of the  $Q_{A,i,j}^\gamma$  and to develop a branes-anti-brane interpretation of the finite  $N$  cutoffs.

## 7 Summary and Outlook

The zero coupling limit of  $\mathcal{N} = 4$  SYM with  $U(N)$  symmetry has enhanced global symmetries. There is a  $U(N)^{\times 4}$  which acts on a complex adjoint scalar field, leaving other fields unchanged, which can be constructed by the standard Noether procedure. This is a symmetry of the classical dilatation operator, which is the Hamiltonian of radial quantisation.

The diagonal  $U(N)$  subgroup of this product is the global gauge symmetry. This leaves physical states invariant. This invariance condition forces the local physical operators to be traces. The full  $U(N)^{\times 4}$  acts non-trivially on the space of gauge invariant operators. By the operator-state correspondence, these gauge invariant operators correspond to physical states. The  $U(N)^{\times 4}$  is a subgroup of  $(U(N^2))^{\times 2}$  which is also present in the theory. These symmetries exist also for the full set of fundamental fields. There are 8 copies of  $U(N)^{\times 4} \subset (U(N^2))^{\times 2}$ , of which 4 on bosonic fields and 4 on fermionic fields. We have used generalised Casimirs constructed from the generators of the enhanced symmetries in order to distinguish the labels on bases for the complete set of operators at finite  $N$  which diagonalise the two-point functions. Some of these Casimirs are rather standard, others are less familiar but nevertheless have the expected invariance properties. Schur-Weyl duality allows the expression of the Casimirs constructed from the Noetherian symmetries in terms of dual algebras including the symmetric group algebra, Brauer algebras and generalisations.

A frequently asked question by students of AdS/CFT is how to see from the string theory of AdS the large  $U(N)$  symmetry of the gauge theory. The usual reply from the teachers is that the  $U(N)$  is not a dynamical symmetry acting on the physical spectrum, so we should not expect to see it in the string theory. This paper revives the question in a refined form. How do we see the charges constructed from the enhanced symmetry  $U(N)^{\times 4}$  from the dual string theory in the limit of  $g_s, \lambda \rightarrow 0$  with  $\frac{\lambda}{g_s} = N$  fixed? Is there a two-dimensional worldsheet formulation in this double scaling limit which captures the finite  $N$  effects such as the cutoffs discussed as the stringy exclusion principle [26, 27, 28] or the enhanced symmetries discussed here, and treats  $g_s N$  as a deformation parameter that breaks the symmetries? This would not be a conventional perturbative string but perhaps something more like Matrix string theory [29] which contains  $N$  as an integer parameter.

The idea of studying the enhanced symmetry points of string theory as an approach that makes optimal use of its hidden symmetries is one that has come up several times in the past, e.g [30]. The hope is that the enhanced symmetries can be used to organise the physics at the special points and also yield information away from the special points. Thanks to the AdS/CFT duality, the zero coupling limit of Yang Mills gives a tractable situation where the power and usefulness of this strategy can be developed, explored and scrutinised for lessons applicable to general string backgrounds. The explicit construction of the worldsheet perturbation expansion has been initiated in [31]. Interesting work on the string theory interpretation of free Yang Mills has also been done in [32].

The simplest of the Casimirs we have considered, which are relevant to the diagonalisation of holomorphic operators have been used in discussions of how to extract detailed information about half-BPS solutions including extremal black holes [15] from the asymptotic fields in the dual spacetime. As explained in section 3.5 we can apply the ideas described here to construct Casimirs which distinguish the complete set of states of a diagonal basis of free  $\mathcal{N} = 4$  SYM. This gives an argument in favour of the idea of integrability of  $\mathcal{N} = 4$  SYM, in a set up that naturally incorporates multi-traces and finite  $N$  effects. As such it supports the view that these charges contain complete information about black holes in  $AdS_5 \times S^5$

spacetime beyond the half-BPS sector. Finding spacetime duals of these enhanced symmetry charges is an important problem. Possibly this would require taking some limit of large quantum numbers where supergravity or semiclassical brane physics can be compared with weak coupling gauge theory. Alternatively a rough qualitative comparison of the number of charges constructed here with those constructible from gravity in spacetime would be interesting.

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## A Casimirs : From $U(N)$ expressions to expressions in terms of finite algebras

In this section, we express Casimirs of the  $U(N)$  group in terms of the symmetric group or the Brauer algebra. This relation essentially comes from the Schur-Weyl duality.

### A.1 $V^{\otimes n}$

Let  $E_j^i$  be the generator of  $u(N)$ . The action of it on the space of the fundamental representation  $V$  is given by

$$E_j^i v^k = \delta_j^k v^i \quad (87)$$

where  $v^i$  denotes the standard basis for the fundamental representation  $V$ . Consider the tensor space  $V^{\otimes n}$ . The action of the generator can be extended to this tensor space as  $\sum_{s=1}^n \rho_s(E)$ , where  $\rho_s(E)$  represents the action on the  $s$ -th space. We also use a notation  $\rho_{\underline{n}}(E) \equiv \sum_{s=1}^n \rho_s(E)$ . The quadratic Casimir on this space can be computed as

$$\begin{aligned} \hat{C}_2 &= \sum_{r,s=1}^n \rho_r(E_j^i) \rho_s(E_i^j) \\ &= \sum_{r=1}^n \rho_r(E_j^i E_i^j) + \sum_{r \neq s} \rho_r(E_j^i) \rho_s(E_i^j) \end{aligned}$$

$$= Nn + \sum_{r \neq s} (rs) \quad (88)$$

Here we have assumed the action on  $v^{i_1} \otimes \cdots \otimes v^{i_n}$ , and the following equations were used.

$$\begin{aligned} (E_j^i E_i^j) v^k &= N v^k \\ (E_j^i \otimes E_i^j) v^{k_r} \otimes v^{k_s} &= v^{k_s} \otimes v^{k_r} = (rs) v^{k_r} \otimes v^{k_s} \end{aligned} \quad (89)$$

On an irreducible space  $p_R V^{\otimes n}$ , we recover a known relation

$$C_2(R) = Nn + 2 \frac{\chi_R(T_2)}{d_R} \quad (90)$$

$T_2$  is a sum of the transpositions.

## A.2 $V^{\otimes m} \otimes \bar{V}^{\otimes n}$

We describe some relevant properties of the Casimirs used in section 3.2. The action of  $u(N)$  on the dual space is

$$E_j^i \bar{v}^r = -\delta^{ir} \bar{v}^j \quad (91)$$

Using this action we verify that  $v^p \bar{v}^p$  is invariant under  $U(N)$ . On the space  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$  the action of  $U(N)$  is dual to the action of the Brauer algebra (Schur-Weyl duality). So we can express  $U(N)$  Casimirs in terms of elements of the Brauer algebra. Calculations are almost parallel to the case of  $V^{\otimes n}$ . In this case we will need

$$\begin{aligned} (E_j^i E_i^j) \bar{v}^k &= N \bar{v}^k \\ (E_j^i \otimes E_i^j) \bar{v}^{k_r} \otimes \bar{v}^{k_s} &= \bar{v}^{k_s} \otimes \bar{v}^{k_r} = (\bar{r}\bar{s}) \bar{v}^{k_r} \otimes \bar{v}^{k_s} \\ (E_j^i \otimes E_i^j) v^{k_r} \otimes \bar{v}^{k_s} &= -\delta^{k_r k_s} \sum_p v^p \otimes \bar{v}^p = -C_{r\bar{s}} v^{k_r} \otimes \bar{v}^{k_s} \end{aligned} \quad (92)$$

$C$  is the contraction which is a linear map from  $V \otimes \bar{V}$  to  $V \otimes \bar{V}$ . The quadratic Casimir on the space  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$  can be computed by

$$\begin{aligned} \hat{C}_2 &= \sum_{r,s=1}^{m+n} \rho_r(E_j^i) \rho_s(E_i^j) \\ &= \sum_{r=1}^{m+n} \rho_r(E_j^i E_i^j) + \sum_{r \neq s} \rho_r(E_j^i) \rho_s(E_i^j) \\ &= (m+n)N + \sum_{r \neq s=1}^m (rs) + \sum_{r \neq s=m+1}^{m+n} (\bar{r}\bar{s}) - 2 \sum_{r=1}^m \sum_{s=m+1}^{m+n} C_{r\bar{s}} \end{aligned} \quad (93)$$

This is a central element of the Brauer algebra.

We next consider another kind of Casimir invariant which plays an important role in this paper. We introduce  $U(N) \times U(N)$ , where the first  $U(N)$  acts on  $V$  and the second one

acts on  $\bar{V}$ . This  $U(N) \times U(N)$  action on  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$  is dual to the action of  $S_m \times S_n$ . For example, we are interested in  $\rho_{\underline{m}}(E_j^i) \rho_{\underline{m}}(E_k^j) \rho_{\underline{n}}(E_i^k)$ , which can be expressed as

$$\begin{aligned}
\hat{C}_{++-} &= \sum_{r,s=1}^m \sum_{t=m+1}^{m+n} \rho_r(E_j^i) \rho_s(E_k^j) \rho_t(E_i^k) \\
&= \sum_{r=1}^m \sum_{t=m+1}^{m+n} \rho_r(E_j^i E_k^j) \rho_t(E_i^k) + \sum_{r \neq s=1}^m \sum_{t=m+1}^{m+n} \rho_r(E_j^i) \rho_s(E_k^j) \rho_t(E_i^k) \\
&= -N \sum_{r=1}^m \sum_{t=m+1}^{m+n} C_{r\bar{t}} - \sum_{r \neq s=1}^m \sum_{t=m+1}^{m+n} (rs) C_{r\bar{t}} \\
&= -N \sum_{r=1}^m \sum_{t=m+1}^{m+n} \left( 1 + \frac{1}{N} \sum_{s(\neq r)=1}^m (rs) \right) C_{r\bar{t}} \tag{94}
\end{aligned}$$

This is an element of the Brauer algebra  $B_N(m, n)$  which is invariant under the subalgebra  $\mathbb{C}(S_m \times S_n)$ . Hence it can be used to distinguish between different multiplicity labels of representations  $A$  of  $\mathbb{C}(S_m \times S_n)$  in an irrep  $\gamma$  of  $B_N(m, n)$ , and is diagonal in the labels  $m_A$  of states in  $A$

$$\langle \gamma, A', m'_{A'}, i | \hat{C}_{++-} | \gamma, A, m_A, j \rangle = \delta_{A'A} \delta_{m_A m'_{A'}} (C_{++-})_{ij} \tag{95}$$

This is demonstrated in section 5. A basis in the multiplicity space can be chosen to diagonalise  $\hat{C}_{++-}$  and this is used in (58).

### A.3 Casimirs for $V^{\otimes(m+n)}$

We describe some relevant properties of the Casimirs used in section 3.3. The quadratic Casimir  $\rho(E_j^i) \rho(E_i^j)$  on the space  $V^{\otimes(m+n)}$  is

$$\begin{aligned}
\hat{C}_2 &= \sum_{r,s=1}^{m+n} \rho_r(E_j^i) \rho_s(E_i^j) \\
&= \sum_{r=1}^{m+n} \rho_r(E_j^i E_i^j) + \sum_{r \neq s=1}^{m+n} \rho_r(E_j^i) \rho_s(E_i^j) \\
&= N(m+n) + \sum_{r \neq s=1}^{m+n} (rs) \tag{96}
\end{aligned}$$

We next consider invariants constructed from  $U(N) \times U(N)$ , where the first  $U(N)$  acts on  $V^{\otimes m}$  and the second one acts on  $V^{\otimes n}$ . The action of the  $U(N) \times U(N)$  is dual to  $S_m \times S_n$ .

$$\hat{C}_{112} = \sum_{r,s=m+1}^{m+n} \sum_{t=1}^m \rho_r(E_j^i) \rho_s(E_k^j) \rho_t(E_i^k)$$

$$\begin{aligned}
&= \sum_{r=m+1}^{m+n} \sum_{t=1}^m \rho_r(E_j^i E_k^j) \rho_t(E_i^k) + \sum_{r \neq s=m+1}^{m+n} \sum_{t=1}^m \rho_r(E_j^i) \rho_s(E_k^j) \rho_t(E_i^k) \\
&= N \sum_{r=m+1}^{m+n} \sum_{t=1}^m (rt) + \sum_{r \neq s=m+1}^{m+n} \sum_{t=1}^m (rs)(rt) \\
&= N \sum_{r=m+1}^{m+n} \sum_{t=1}^m \left( 1 + \frac{1}{N} \sum_{s(\neq r)=m+1}^{m+n} (rs) \right) (rt)
\end{aligned} \tag{97}$$

This is an element of  $\mathbb{C}(S_{m+n})$  which is invariant under the subalgebra  $\mathbb{C}(S_m \times S_n)$ . It is diagonal in the state labels of the subalgebra, and mixes the multiplicity labels (see equation (64)). A diagonalising basis in the multiplicity space can be chosen. This is demonstrated in section 5 and is used in section 4.2.

#### A.4 $U(N^2)$

Here we express Casimirs constructed  $U(N^2)$  along with  $U(N) \times U(N)$ , in terms of the symmetric groups.

Let  $E_{lm}^{jk}$  be the generator of  $U(N^2)$

$$E_{lm}^{jk} v_q^p = \delta_m^p \delta_q^j v_l^k \tag{98}$$

and  $(E_{\mathcal{L}})_j^l$  and  $(E_{\mathcal{R}})_k^m$  be the generators of the left and the right action of  $U(N)$

$$\begin{aligned}
(E_{\mathcal{L}})_j^l v_q^p &= \delta_j^p v_q^l \\
(E_{\mathcal{R}})_k^m v_q^p &= \delta_q^m v_k^p
\end{aligned} \tag{99}$$

The quadratic Casimirs of these actions of  $U(N)$  are

$$\begin{aligned}
\hat{C}_2^{\mathcal{L}} &= \sum_{r,s=1}^n \rho_r((E_{\mathcal{L}})_j^i) \rho_s((E_{\mathcal{L}})_i^j) = Nn + \sum_{r \neq s} (rs)_{\mathbf{u}} \\
\hat{C}_2^{\mathcal{R}} &= \sum_{r,s=1}^n \rho_r((E_{\mathcal{R}})_j^i) \rho_s((E_{\mathcal{R}})_i^j) = Nn + \sum_{r \neq s} (rs)_{\mathbf{l}}
\end{aligned} \tag{100}$$

The calculations are completely similar to (88).  $(rs)_{\mathbf{u}}$  and  $(rs)_{\mathbf{l}}$  are the actions on the upper and lower indices of  $v_{q_r}^{p_r} \otimes v_{q_s}^{p_s}$ , respectively.

The quadratic Casimir for  $U(N^2)$  can be computed as

$$\begin{aligned}
\sum_{r,s=1}^n \rho_r(E_{lm}^{jk}) \rho_s(E_{jk}^{lm}) &= \sum_{r=1}^n \rho_r(E_{lm}^{jk} E_{jk}^{lm}) + \sum_{r \neq s} \rho_r(E_{lm}^{jk}) \rho_s(E_{jk}^{lm}) \\
&= \sum_{r \neq s} (rs)_{\mathbf{l}} (rs)_{\mathbf{u}} + N^2 n
\end{aligned} \tag{101}$$



We are interested in invariants such as  $\rho(E_{lm}^{jk})\rho((E_{\mathcal{L}})_k^m)\rho((E_{\mathcal{R}})_j^l)$  which are constructed from  $U(N^2)$  generators and which is invariant under the  $U(N) \times U(N)$  subalgebra. It is also expressed in terms of the symmetric groups as

$$\begin{aligned}
& \sum_{r,s,t=1}^n \rho_r(E_{lm}^{jk})\rho_s((E_{\mathcal{L}})_k^m)\rho_t((E_{\mathcal{R}})_j^l) \\
&= \sum_{r \neq s \neq t} \rho_r(E_{lm}^{jk})\rho_s((E_{\mathcal{L}})_k^m)\rho_t((E_{\mathcal{R}})_j^l) + \sum_{r \neq t} \rho_r(E_{lm}^{jk}(E_{\mathcal{L}})_k^m)\rho_t((E_{\mathcal{R}})_j^l) \\
&\quad + \sum_{r \neq s} \rho_r(E_{lm}^{jk}(E_{\mathcal{R}})_j^l)\rho_s((E_{\mathcal{L}})_k^m) + \sum_r \rho_r(E_{lm}^{jk}(E_{\mathcal{L}})_k^m(E_{\mathcal{R}})_j^l) \\
&= \sum_{r \neq s \neq t} (rs)_{\mathbf{u}}(rt)_1 + N \sum_{r \neq s} (rs)_1 + N \sum_{r \neq s} (rs)_{\mathbf{u}} + \sum_{r \neq s} (rs)_1(rs)_{\mathbf{u}} + N^2 n \quad (102)
\end{aligned}$$

For example, the first term was calculated as

$$\begin{aligned}
& (E_{lm}^{jk} \otimes (E_{\mathcal{L}})_k^m \otimes (E_{\mathcal{R}})_j^l) v_{q_r}^{p_r} \otimes v_{q_s}^{p_s} \otimes v_{q_t}^{p_t} \\
&= v_{q_t}^{p_s} \otimes v_{q_s}^{p_r} \otimes v_{q_r}^{p_t} \\
&= (rs)_{\mathbf{u}}(rt)_1 v_{q_r}^{p_r} \otimes v_{q_s}^{p_s} \otimes v_{q_t}^{p_t} \quad (103)
\end{aligned}$$

This leads to the result (69). The expression in the last line of (102) is in  $\mathbb{C}(S_n) \times \mathbb{C}(S_n)$  and is invariant under the diagonal subalgebra  $\mathbb{C}(S_n)$ . Such elements can distinguish multiplicity labels for  $S_n$  irreps.  $\Lambda$  in the tensor products  $R_1 \otimes R_2$ . This is illustrated in section 5 and used in 4.3.

## B $Q$ -operators and the algebraic Brauer construction of matrix units

In this section, we give an algebraic expression of the symmetric branching operator, which was proposed to give a complete set of gauge invariant operators constructed from  $X$  and  $X^\dagger$  [9]. An introduction of the Brauer algebra is given in the section 3 in [9].

Representation matrix elements of a Brauer algebra element  $b$  are denoted as

$$D_{IJ}^\gamma(b) = \langle \gamma, I | b | \gamma, J \rangle \quad 1 \leq I, J \leq d_\gamma \quad (104)$$

where  $I$  labels states in an irreducible representation  $\gamma$  of the Brauer algebra, and  $d_\gamma$  is the dimension of  $\gamma$ . The character is given by the trace of the matrix element

$$\sum_I D_{II}^\gamma(b) = \sum_I \langle \gamma, I | b | \gamma, I \rangle = \chi_\gamma(b) \quad (105)$$

We define

$$P_{IJ}^\gamma \equiv t_\gamma \sum_i D_{JI}^\gamma(b_i) b_i^*, \quad (106)$$

which satisfies [33]

$$P_{IJ}^\gamma P_{KL}^{\gamma'} = \delta^{\gamma\gamma'} \delta_{JK} P_{IL}^\gamma. \quad (107)$$

$b^*$  is defined as a dual element of  $b$  by  $tr_{m,n}(bb^*) = 1$ . For more details, see [33] or the section 3 in [9]. The trace of  $P_{IJ}^\gamma$  gives the central Brauer projector

$$P^\gamma = \sum_I P_{II}^\gamma = t_\gamma \sum_i \chi^\gamma(b_i) b_i^*. \quad (108)$$

The  $P_{IJ}^\gamma$  are elements of the algebra called *matrix units*. They can be identified with  $|\gamma, I\rangle\langle\gamma, J|$ . The  $b_i^*$  is defined as the dual under a trace, i.e  $tr(b_i b_j^*) = \delta_{ij}$  [33]. When the trace is in  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ , the normalisation factor is  $t_\gamma = Dim\gamma$ , which is the dimension of the  $U(N)$  representation corresponding to the Young diagram  $\gamma$ , having positive row lengths adding to  $m$  and negative row lengths adding to  $n$ .

The matrix unit can be used to get a representation matrix from a character as

$$\begin{aligned} \chi^\gamma(P_{IJ}^{\gamma'} b) &= t^{\gamma'} \sum_{b'} D_{JI}^{\gamma'}(b') \chi^\gamma(b'^* b) \\ &= t^{\gamma'} \sum_{b'} D_{JI}^{\gamma'}(b') D_{KL}^\gamma(b'^*) D_{LK}^\gamma(b) \\ &= D_{JI}^\gamma(b) \delta^{\gamma\gamma'} \end{aligned} \quad (109)$$

where we have used the orthogonality of representation matrices for the Brauer algebra

$$\sum_b D_{IJ}^\gamma(b) D_{KL}^{\gamma'}(b^*) = \frac{1}{t^\gamma} \delta_{JK} \delta_{IL} \quad (110)$$

In our previous paper [9], we have defined the  $Q$ -operator (symmetric branching operator) which has the following properties

$$Q_{A,ij}^\gamma Q_{B,kl}^{\gamma'} = \delta_{\gamma\gamma'} \delta_{AB} \delta_{jk} Q_{A,il}^\gamma \quad (111)$$

$$tr_{m,n}(Q_{A,ij}^\gamma) = \delta_{ij} d_A Dim\gamma \quad (112)$$

$$h Q_{A,ij}^\gamma h^{-1} = Q_{A,ij}^\gamma \quad h \in \mathbb{C}(S_m \times S_n) \quad (113)$$

$A$  labels an irreducible representation of  $\mathbb{C}(S_m \times S_n)$ . The indices  $i, j$  are labels to distinguish different copies of the representation  $A$  in the representation  $\gamma$ .

We now introduce the branching coefficient as

$$B_{\gamma,I;A,m_A,i} \equiv \langle\gamma, I | \gamma \rightarrow A, m_A, i\rangle \quad (114)$$

The basis  $|\gamma \rightarrow A, m_A, i\rangle$  represents the decomposition of  $\gamma$  in terms of the subalgebra  $\mathbb{C}(S_m \times S_n)$ , and  $m_A$  labels states in  $A$  obeying  $0 \leq m_A \leq d_A$ . We can always choose an orthogonal basis

$$\langle\gamma \rightarrow B, m_B, k | \gamma \rightarrow A, m_A, j\rangle = \delta_{AB} \delta_{m_A m_B} \delta_{jk} \quad (115)$$

Using this orthogonality and the completeness of  $|\gamma, I\rangle$ , we can show

$$\begin{aligned} \sum_I (B_{\gamma, I; A, m_A, i_A})^\dagger B_{\gamma, I; B, m_B, i_B} &= \sum_I \langle \gamma \rightarrow A, m_A, i_A | \gamma, I \rangle \langle \gamma, I | \gamma \rightarrow B, m_B, i_B \rangle \\ &= \delta_{AB} \delta_{m_A m_B} \delta_{i_A i_B} \end{aligned} \quad (116)$$

The operator  $Q_{A, ij}^\gamma$  can be expressed in terms of the operator (106) and the branching coefficient (114) as

$$Q_{A, ij}^\gamma = \sum_{m_A, I, J} B_{\gamma I; A, m_A, i}^\dagger P_{IJ}^\gamma B_{\gamma J; A, m_A, j} \quad (117)$$

This expresses  $Q_{A, ij}^\gamma$  in terms of Brauer algebra elements. It is easy to show (111) if we use the orthogonality of the branching coefficient (116). (112) can be shown as

$$\begin{aligned} \text{tr}_{m, n}(Q_{A, ij}^\gamma) &= \sum_{I, J} \text{tr}_{m, n}(P_{IJ}^\gamma) \sum_{m_A} B_{\gamma I; A, m_A, i}^\dagger B_{\gamma J; A, m_A, j} \\ &= \text{Dim} \gamma \sum_{m_A, I} B_{\gamma I; A, m_A, i}^\dagger B_{\gamma I; A, m_A, j} \\ &= \text{Dim} \gamma \delta_{ij} \sum_{m_A} \delta_{m_A m_A} \\ &= \text{Dim} \gamma d_A \delta_{ij} \end{aligned} \quad (118)$$

To show the second line we have used  $\text{tr}_{m, n}(P_{IJ}^\gamma) = \text{Dim} \gamma \delta_{IJ}$  which follows from the Schur-Weyl duality for  $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ .

Another useful property of  $Q_{A, ij}^\gamma$  is

$$(b) Q_{A, ij}^\gamma = \sum_k C_{A, ki}^\gamma(b) Q_{A, kj}^\gamma = \frac{1}{d_A} \sum_{m_A, k} \langle \gamma, A, m_A, k | (b) | \gamma, A, m_A, i \rangle Q_{A, kj}^\gamma \quad (119)$$

Here  $(b)$  is an element which commutes with any element in  $\mathbb{C}(S_m \times S_n)$ . This uses a related property of  $P_{IJ}^\gamma$

$$b P_{IJ}^\gamma = \sum_K D_{KI}^\gamma(b) P_{KJ}^\gamma \quad (120)$$

## B.1 Restricted characters for Brauer algebra

Let us define the *restricted character* for the Brauer algebra

$$\begin{aligned} \chi_{A, ij}^\gamma(b) &\equiv \sum_{m_A, I, J} B_{\gamma I; A, m_A, i}^\dagger D_{JI}^\gamma(b) B_{\gamma J; A, m_A, j} \\ &= \sum_{m_A, I, J} \langle \gamma \rightarrow A, m_A, i | \gamma, I \rangle \langle \gamma, I | b | \gamma, J \rangle \langle \gamma, J | \gamma \rightarrow A, m_A, j \rangle \\ &= \sum_{m_A} \langle \gamma \rightarrow A, m_A, i | b | \gamma \rightarrow A, m_A, j \rangle \end{aligned} \quad (121)$$

The expression given by (117) can be re-written as

$$\begin{aligned} Q_{A,ij}^\gamma &= \sum_{m_A} B_{\gamma I;A,m_A,i}^\dagger P_{IJ}^\gamma B_{\gamma J;A,m_A,j} \\ &= t_\gamma \sum_b \chi_{A,ij}^\gamma(b) b^* \end{aligned} \quad (122)$$

The terminology “restricted character” has been used for analogous objects constructed in the reduction of  $S_{m+n}$  to  $S_m \times S_n$  in the context of strings attached to giant gravitons. We will elaborate on this below. Conversely the restricted characters can be expressed in terms of the  $Q$  operators

$$\chi_{A,ij}^\gamma(b) = \chi^\gamma(Q_{A,ji}^\gamma b) \quad (123)$$

This can be easily derived using (109) and (117). We can also show

$$\chi^\gamma(b) = \sum_{A,i} \chi_{A,ii}^\gamma(b) \quad (124)$$

The following expression which is isomorphic to the algebraic expression is also useful to understand relations we have derived

$$Q_{A,ij}^\gamma = \sum_m |\gamma \rightarrow A, m, i\rangle \langle \gamma \rightarrow A, m, j| \quad (125)$$

## B.2 Symmetric elements in terms of Matrix units.

Since  $\sum_{h \in S_m \times S_n} h b h^{-1}$  commutes with any elements in the subalgebra  $\mathbb{C}(S_m \times S_n)$ , it can be expressed as a linear combination of  $Q_{A,ij}^\gamma$ .

$$(b) \equiv \frac{1}{m!n!} \sum_{h \in S_m \times S_n} h b h^{-1} = C_{A,ij(b)}^\gamma Q_{A,ij}^\gamma \quad (126)$$

Multiplying by  $Q_{B,kl}^{\gamma'}$  on both sides of (126) and using

$$\begin{aligned} Q_{A,ij}^\gamma Q_{B,kl}^{\gamma'} &= \delta_{\gamma\gamma'} \delta_{AB} \delta_{jk} Q_{A,il}^\gamma \\ \text{tr}(Q_{A,ij}^\gamma) &= \delta_{ij} d_A \text{Dim} \gamma \end{aligned} \quad (127)$$

we find the coefficient  $C_{A,ij(b)}^\gamma$  can be obtained as

$$\begin{aligned} C_{A,ij(b)}^\gamma &= \frac{1}{d_A \text{Dim} \gamma} \text{tr}_{m,n}((b) Q_{A,ji}^\gamma) \\ &= \frac{1}{d_A \text{Dim} \gamma} \sum_{\gamma'} \text{Dim} \gamma' \chi_{\gamma'}((b) Q_{A,ji}^\gamma) \\ &= \frac{1}{d_A} \chi_\gamma((b) Q_{A,ji}^\gamma) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{d_A} \chi_{A,ij}^\gamma((b)) \\
&= \frac{1}{d_A} \chi_{A,ij}^\gamma(b)
\end{aligned} \tag{128}$$

We have used the Schur-Weyl duality to get the second line, and the following property for the restricted character for the Brauer algebra defined in (122) has been used to derive the last equality

$$\chi_{A,ij}^\gamma(hbh^{-1}) = \chi_{A,ij}^\gamma(b), \quad h \in \mathbb{C}(S_m \times S_n) \tag{129}$$

### B.3 Restricted Schur Operators for $X, X^\dagger$

The restricted Schur operators of [34, 35, 36, 37] can be described using the language developed above. This highlights the similarities between the Brauer construction and the symmetric group construction, explained in broad outline at the beginning of section 4.2.

Analogous to (106) the matrix units for  $S_{m+n}$  are

$$P_{IJ}^R = \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} D_{JI}^R(\sigma) \sigma^{-1} = \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} D_{IJ}^R(\sigma) \sigma \tag{130}$$

where we used  $D_{JI}^R(\sigma^{-1}) = D_{IJ}^R(\sigma)$ . The analog of the  $Q_{\alpha,\beta;i,j}^\gamma$  is  $Q_{R_1,R_2;i,j}^R$  which can be written as

$$Q_{R_1,R_2;i,j}^R = B_{R_1,R_2,m_{R_1},m_{R_2},i;R,I} P_{IJ}^R B_{R,J;R_1,R_2,m_{R_1},m_{R_2},j} \tag{131}$$

The construction of gauge invariant operators is done as  $\text{tr}_{m+n}(Q_{R_1,R_2;i,j}^R(\mathbf{X} \otimes \mathbf{X}^\dagger))$ . Restricted schur operators were investigated in [38]. A similar formula to (126) was given.

## C Vershik-Okounkov approach

In this section, we review the Vershik-Okounkov approach [39] to the representation theory of the symmetric group, which we use in section 5.

For the group algebra of the symmetric group  $S_n$ , define

$$X_i = (1, i) + (2, i) + \cdots + (i-1, i) \tag{132}$$

for  $i = 2, \dots, n$  and  $X_1 = 0$ . This set gives us maximally commuting elements of the group algebra. They are called the Young-Jucys-Murphy elements (YJM-elements).

In every irreducible representation, the eigenvalues of the  $X_i$  uniquely choose a basis. Let  $v$  be a basis vector in an irrep. We denote the vector of eigenvalues by

$$\alpha(v) = (a_1, a_2, \dots, a_n) \tag{133}$$

where  $a_i$  is the eigenvalue of  $X_i$  on  $v$  ( $a_1 = 0$  due to  $X_1 = 0$ ). It is a non-trivial result that the  $a_i$  can be read off from standard Young tableaux. Indeed  $a_i$  is equal to the content of a box containing the number  $i$  in a standard Young tableau. The content of a box is its  $x$ -coordinate minus its  $y$ -coordinate, where  $x$ -axis is drawn from left to right and  $y$ -axis is from top to bottom. The content is shown in figure 1.

0	1	2	3	...
-1	0	1		
-2				
⋮				

Figure 1: Contents of boxes in Young tableau

Let us show some examples. The first example is the  $[2, 1]$  representation of  $S_3$ . As shown in figure 2, there are two standard tableaux, which means the dimension of this representation is 2. For a representation,  $X_i$  are given by

$$\begin{aligned} X_2 &= \text{diag}(-1, 1) \\ X_3 &= \text{diag}(1, -1) \end{aligned} \quad (134)$$

Basis vectors  $e_1 = (1, 0)^T$  and  $e_2 = (0, 1)^T$  correspond to the first and the second standard tableaux shown in figure 2 respectively.

1	3
2	

1	2
3	

Figure 2: two standard tableaux for  $\lambda = [2, 1]$  of  $S_3$

The second example is the  $[3, 2]$  representation of  $S_5$ .  $X_i$  are given by

$$\begin{aligned} X_2 &= \text{diag}(-1, 1, -1, 1, 1) \\ X_3 &= \text{diag}(1, -1, 1, -1, 2) \\ X_4 &= \text{diag}(0, 0, 2, 2, -1) \\ X_5 &= \text{diag}(2, 2, 0, 0, 0) \end{aligned} \quad (135)$$

Basis vectors  $e_i$  ( $i = 1, \dots, 5$ ) correspond to the  $i$ -th tableau given in figure 3.

The last example is the  $[2, 2, 1]$  representation of  $S_5$ .  $X_i$  are given by

$$\begin{aligned} X_2 &= \text{diag}(-1, -1, 1, -1, 1) \\ X_3 &= \text{diag}(-2, 1, -1, 1, -1) \\ X_4 &= \text{diag}(1, -2, -2, 0, 0) \\ X_5 &= \text{diag}(0, 0, 0, -2, -2) \end{aligned} \quad (136)$$

Basis vectors  $e_i$  ( $i = 1, \dots, 5$ ) correspond to the  $i$ -th tableau given in figure 4.



$(13)t_{\tau_1} = t_{\tau_2}$ ,  $(13)t_{\tau_2} = t_{\tau_1}$ . Then we obtain the following representation matrices

$$(12) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad (23) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (140)$$

We next consider the Brauer algebra  $B_N(m, n)$ . The irreducible representation of  $B_N(m, n)$  is indexed by a sequence of integers  $\gamma = (\gamma_1, \dots, \gamma_N)$  obeying  $\gamma_1 \geq \dots \geq \gamma_N$ . This sequence of integers is called  $N$ -staircase. Because the positive integers of the sequence determine a partition, a staircase contains two partitions  $\gamma^+$  and  $\gamma^-$ , which come from the positive and negative parts of  $\gamma$  respectively. If  $\gamma^+$  and  $\gamma^-$  are given by a partition of  $m - k$  and that of  $n - k$  respectively, choosing a staircase  $\gamma$  is equivalent to choosing a set of  $(\gamma^+, \gamma^-, k)$ , where  $k$  is an integer with  $0 \leq k \leq \min(m, n)$ . It follows from these definitions that  $c_1(\gamma^+) + c_1(\gamma^-) \leq N$ , where  $c_1(\gamma^+)$  is the length of the first column of  $\gamma^+$ . In this section, we assume this condition is always satisfied.

We introduce two set of integers  $\mathcal{P} = \{1, 2, \dots, m\}$  and  $\mathcal{Q} = \{m + 1, \dots, m + n\}$ . Let  $\underline{p} = (p_1, \dots, p_k)$  and  $\underline{q} = (q_1, \dots, q_k)$  be ordered subsets of  $\mathcal{P}$  and  $\mathcal{Q}$ , and  $\underline{p}^c$  and  $\underline{q}^c$  be complements of  $\underline{p}$  and  $\underline{q}$  in  $\mathcal{P}$  and  $\mathcal{Q}$ . We define a set  $(\tau, \underline{p}, \underline{q})$  of standard tableau  $\tau$  and ordered subsets of integers  $\underline{p}$  and  $\underline{q}$ .  $\tau = [\tau^+, \tau^-]$  is called standard if both  $\tau^+$  and  $\tau^-$  are standard.  $\tau^+$  ( $\tau^-$ ) contains numbers in  $\underline{p}^c$  ( $\underline{q}^c$ ). The dimension of irreducible representations of Brauer algebra is equivalent to the number of the set  $(\tau, \underline{p}, \underline{q})$ . For example, in the irreducible representation  $\gamma = (k = 2, [1], [1])$  of  $B(3, 3)$ , there are 18 sets, which are given by

$$\begin{aligned} \tau = [1, 4] & \quad \text{with } \underline{p} = (2, 3) \text{ and } \underline{q} = (5, 6) \\ \tau = [1, 4] & \quad \text{with } \underline{p} = (2, 3) \text{ and } \underline{q} = (6, 5) \\ \tau = [1, 5] & \quad \text{with } \underline{p} = (2, 3) \text{ and } \underline{q} = (4, 6) \\ \tau = [1, 5] & \quad \text{with } \underline{p} = (2, 3) \text{ and } \underline{q} = (6, 4) \\ \tau = [1, 6] & \quad \text{with } \underline{p} = (2, 3) \text{ and } \underline{q} = (4, 5) \\ \tau = [1, 6] & \quad \text{with } \underline{p} = (2, 3) \text{ and } \underline{q} = (5, 4) \\ \tau = [2, 4] & \quad \text{with } \underline{p} = (1, 3) \text{ and } \underline{q} = (5, 6) \\ \tau = [2, 4] & \quad \text{with } \underline{p} = (1, 3) \text{ and } \underline{q} = (6, 5) \\ \tau = [2, 5] & \quad \text{with } \underline{p} = (1, 3) \text{ and } \underline{q} = (4, 6) \\ \tau = [2, 5] & \quad \text{with } \underline{p} = (1, 3) \text{ and } \underline{q} = (6, 4) \\ \tau = [2, 6] & \quad \text{with } \underline{p} = (1, 3) \text{ and } \underline{q} = (4, 5) \\ \tau = [2, 6] & \quad \text{with } \underline{p} = (1, 3) \text{ and } \underline{q} = (5, 4) \\ \tau = [3, 4] & \quad \text{with } \underline{p} = (1, 2) \text{ and } \underline{q} = (5, 6) \\ \tau = [3, 4] & \quad \text{with } \underline{p} = (1, 2) \text{ and } \underline{q} = (6, 5) \\ \tau = [3, 5] & \quad \text{with } \underline{p} = (1, 2) \text{ and } \underline{q} = (4, 6) \\ \tau = [3, 5] & \quad \text{with } \underline{p} = (1, 2) \text{ and } \underline{q} = (6, 4) \\ \tau = [3, 6] & \quad \text{with } \underline{p} = (1, 2) \text{ and } \underline{q} = (4, 5) \\ \tau = [3, 6] & \quad \text{with } \underline{p} = (1, 2) \text{ and } \underline{q} = (5, 4) \end{aligned}$$



where we have introduced an abbreviation  $\tau = [1, 4]$ , which means the following Young tableau

$$\tau = \left[ \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline \end{array} \right]$$

For a given set of  $(\tau, \underline{p}, \underline{q})$ , linearly independent basis  $t_{\tau, \underline{p}, \underline{q}}$  are given by the following.

$$\beta_{\tau, \underline{p}, \underline{q}} = u_1 \otimes \cdots \otimes u_m \otimes u_{m+1}^* \otimes \cdots \otimes u_{m+n}^* \quad (141)$$

is a tensor whose factors are defined by

$$u_l = \begin{cases} v_1 & \text{if } l \in \underline{p} \\ v_j & \text{if } l \in \underline{p}^c \text{ and } l \text{ in the } j\text{-th row of } \tau^+ \end{cases}$$

$$u_l^* = \begin{cases} v_1^* & \text{if } l \in \underline{q} \\ v_{N-j+1}^* & \text{if } l \in \underline{q}^c \text{ and } l \text{ in the } j\text{-th row of } \tau^- \end{cases}$$

From this,  $t_{\tau, \underline{p}, \underline{q}}$  is defined by

$$t_{\tau, \underline{p}, \underline{q}} = y_\tau C_{\underline{p}, \underline{q}} \beta_{\tau, \underline{p}, \underline{q}} \quad (142)$$

where

$$C_{\underline{p}, \underline{q}} = C_{p_1, q_1} \cdots C_{p_k, q_k} \quad (143)$$

is the product of  $k$  contractions.  $C_{p, q}$  is the contraction acting on the  $p$ -th factor and the  $q$ -th factor as  $C v_i \otimes v_j = \delta_{ij} \sum_l v_l \otimes v_l$ .  $y_\tau = y_{\tau^+} y_{\tau^-}$  is the Young symmetriser. For the set of  $(\tau, \underline{p}, \underline{q})$  of  $\gamma = (k = 2, [1], [1])$ , the bases are given by

$$\begin{aligned} t^{(1)} &= C_{25} C_{36} \beta_{\tau[1,4]} & t^{(2)} &= C_{35} C_{26} \beta_{\tau[1,4]} \\ t^{(3)} &= C_{24} C_{36} \beta_{\tau[1,5]} & t^{(4)} &= C_{34} C_{26} \beta_{\tau[1,5]} \\ t^{(5)} &= C_{24} C_{35} \beta_{\tau[1,6]} & t^{(6)} &= C_{34} C_{25} \beta_{\tau[1,6]} \\ t^{(7)} &= C_{15} C_{36} \beta_{\tau[2,4]} & t^{(8)} &= C_{16} C_{35} \beta_{\tau[2,4]} \\ t^{(9)} &= C_{14} C_{36} \beta_{\tau[2,5]} & t^{(10)} &= C_{16} C_{34} \beta_{\tau[2,5]} \\ t^{(11)} &= C_{14} C_{35} \beta_{\tau[2,6]} & t^{(12)} &= C_{15} C_{34} \beta_{\tau[2,6]} \\ t^{(13)} &= C_{15} C_{26} \beta_{\tau[3,4]} & t^{(14)} &= C_{16} C_{25} \beta_{\tau[3,4]} \\ t^{(15)} &= C_{14} C_{26} \beta_{\tau[3,5]} & t^{(16)} &= C_{16} C_{24} \beta_{\tau[3,5]} \\ t^{(17)} &= C_{14} C_{25} \beta_{\tau[3,6]} & t^{(18)} &= C_{15} C_{24} \beta_{\tau[3,6]} \end{aligned} \quad (144)$$

Some examples of actions with the permutations and the contractions on these states are

$$\begin{aligned} (12)t^{(1)} &= C_{15} C_{36} \beta_{\tau[2,4]} = t^{(7)} \\ C_{14}t^{(1)} &= C_{14} C_{25} C_{36} \beta_{\tau[1,4]} = C_{25} C_{36} C_{14} \beta_{\tau[1,4]} = 0 \\ C_{14}t^{(3)} &= C_{14} C_{24} C_{36} \beta_{\tau[1,5]} = (12)C_{24} C_{36} \beta_{\tau[1,5]} = C_{14} C_{36} \beta_{\tau[2,5]} = t^{(9)} \end{aligned}$$

$$C_{14}t^{(9)} = C_{14}C_{14}C_{36}\beta_{\tau[2,5]} = NC_{14}C_{36}\beta_{\tau[2,5]} = Nt^{(9)} \quad (145)$$

These are calculated using relations such as

$$\begin{aligned} C_{i\bar{j}}C_{i\bar{k}} &= C_{i\bar{j}}(\bar{j}\bar{k}) = (\bar{j}\bar{k})C_{i\bar{k}} \\ C_{i\bar{j}}C_{k\bar{j}} &= C_{i\bar{j}}(ik) = (ik)C_{k\bar{j}} \\ C_{i\bar{j}}C_{i\bar{j}} &= NC_{i\bar{j}} \end{aligned} \quad (146)$$

Then representations for (12), (13), (45) and (46) are obtained as

$$\begin{aligned} (12) &= F_{1,7} + F_{2,8} + F_{3,9} + F_{4,10} + F_{5,11} + F_{6,12} + F_{13,14} + F_{15,16} + F_{17,18} \\ (13) &= F_{1,14} + F_{2,13} + F_{3,16} + F_{4,15} + F_{5,18} + F_{6,17} + F_{7,8} + F_{9,10} + F_{11,12} \\ (45) &= F_{1,3} + F_{2,4} + F_{5,6} + F_{7,9} + F_{8,10} + F_{11,12} + F_{13,15} + F_{14,16} + F_{17,18} \\ (46) &= F_{1,6} + F_{2,5} + F_{3,4} + F_{7,12} + F_{8,11} + F_{9,10} + F_{13,18} + F_{14,17} + F_{15,16} \end{aligned} \quad (147)$$

Here we have introduced  $F_{i,j} = E_{i,j} + E_{j,i}$ , where  $E_{i,j}$  is a matrix whose nonzero component is only  $(i, j)$  component. (23) and (56) can be computed by (23) = (12)(13)(12) and (56) = (45)(46)(45).

The contraction  $C_{14}$  is calculated as

$$\begin{aligned} C_{14} &= E_{9,3} + E_{15,4} + E_{11,5} + E_{17,6} + E_{9,7} + E_{11,8} + NE_{9,9} + E_{9,10} + NE_{11,11} \\ &\quad + E_{11,12} + E_{15,13} + E_{17,14} + NE_{15,15} + E_{15,16} + NE_{17,17} + E_{17,18} \end{aligned} \quad (148)$$

Other contractions such as  $C_{15}$  or  $C_{26}$  can be calculated by

$$C_{ij} = (i1)(4j)C_{14}(i1)(4j) \quad (149)$$

where  $i = 1, 2, 3$  and  $j = 4, 5, 6$ .

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